Lecture 1: Defect Operators in Geometric Engineering

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Abstract: We introduce the defect group, focussing on string constructed systems, and discuss its utility.

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1 Introduction and Motivating Examples

In this lecture we introduce the defect group. We focus on non-gravitational systems with extra-dimensional constructions in string theory and utilize the extra-dimensional geometry in computing and describing the defect group of such systems. Looking ahead, we will find the defect group to furnish representations of the systems generalized global symmetries. Further, the defect group will motivate the top down construction of non-invertible symmetry operators and symmetry theories, including symmetry topological field theories.

There are by now many lecture notes on generalized global symmetries [1–6]. Complementarily to these, we will focus here mainly on top down and largely geometric perspectives, and highlight their advantages and applications. With the hope of giving an accessible overview on these constructions we will proceed mostly by way of example, focussing first on situations where the field theoretic discussion is simple and well-understood.

1.1 Defects in Gauge Theories

We begin with a field theoretic discussion of defects in gauge theories preparing first extra-dimensional analyses which will map onto the introduced structures. First, we focus on 4D Yang-Mills theory then we briefly comment on natural generalizations to arbitrary dimensions. In all cases we present well-known screening arguments, see for example [7].

1.1.1 Electric and Magnetic Line Defects in 4D Yang-Mills theory

Consider 4D Yang-Mills theory with simply-laced gauge algebra $\mathfrak{g}$ of type ADE in Cartan's classification and action

$$S[\mathfrak{g}] = -\frac{1}{4g_{\text{YM}}^2}\int \text{Tr}(F\wedge *F)\,, \tag{1.1}$$

with field strength $F = dA + A\wedge A$ and connection 1-form A valued in the adjoint representation of the Lie algebra $\mathfrak{g}$. Pick a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$. Then, diagonalizing the adjoint action, one obtains roots $\Phi(\mathfrak{g})\subset\mathfrak{h}^*$. The roots generate the root lattice $\Lambda_{\text{root}}(\mathfrak{g})$ with pairing set by the Killing form. Dynamical excitations carry electric charges valued in the root lattice. Similarly, magnetic charges are valued in the root lattice $\Lambda_{\text{root}}(\mathfrak{g}^\vee) = \Lambda_{\text{coroot}}(\mathfrak{g})$ where $\mathfrak{g}^\vee$ is the GNO dual Lie algebra. Overall, the charge lattice of the dynamical excitations is

$$\Lambda(\mathfrak{g}) = \Lambda_{\text{root}}(\mathfrak{g})\oplus\Lambda_{\text{coroot}}(\mathfrak{g})\,. \tag{1.2}$$

Wilson lines and 't Hooft lines are in contrast labelled by elements of the weight and coweight lattices $\Lambda_{\text{weight}}(\mathfrak{g}), \Lambda_{\text{coweight}}(\mathfrak{g})$ respectively. They are non-dynamical sources, however due to Gluons being in the adjoint representation, there is no Gauss law measuring the precise representation associated with weights and coweights. Rather, the conserved quantity that can be measured is the N-ality of the line operator as captured by

$$\mathbb{D}^{(1)}(\mathfrak{g}) \equiv \frac{\Lambda_{\text{weight}}(\mathfrak{g})}{\Lambda_{\text{root}}(\mathfrak{g})}\oplus\frac{\Lambda_{\text{coweight}}(\mathfrak{g})}{\Lambda_{\text{coroot}}(\mathfrak{g})}\cong \mathcal{Z}_{\text{elec}}(\mathfrak{g})\oplus\mathcal{Z}_{\text{mag}}(\mathfrak{g})\,, \tag{1.3}$$

referred to as the defect group of line operators or simply the defect group of lines. The defect group of lines is a finite group. It evaluates according to $\mathcal{Z}_{\text{elec}}(\mathfrak{g})\cong\mathcal{Z}_{\text{mag}}(\mathfrak{g})\cong\mathcal{Z}(\mathfrak{g})$ where $\mathcal{Z}(\mathfrak{g})$ is the center of the simply connected Lie group with Lie algebra $\mathfrak{g}$. For example, one hase $\mathcal{Z}(\mathfrak{su}(N))\cong\mathbb{Z}_N$ the center of $SU(N)$. It comes with a pairing determined by the Dirac pairing

$$\langle\,\cdot\,,\cdot\,\rangle\,:\,\mathbb{D}^{(1)}(\mathfrak{g})\times\mathbb{D}^{(1)}(\mathfrak{g})\to\mathbb{Q}/\mathbb{Z}\,,\qquad \langle(e,m),(e',m')\rangle = \exp\left(\frac{2\pi i}{|\mathcal{Z}(\mathfrak{g})|}(em'-e'm)\right)\,, \tag{1.4}$$

which determines the mutual non-locality between elements of $\mathbb{D}^{(1)}$. Here we assumed that $\mathcal{Z}(\mathfrak{g})$ is cyclic. The defect group of lines is a property of the relative theory, in the sense of [8], labelled by the Lie algebra $\mathfrak{g}$.

Maximally mutually local subsets $\mathcal{P}\subset\mathbb{D}^{(1)}$ with respect to the pairing $\langle\,\cdot\,,\cdot\,\rangle$ determine the gauge group $G_\mathcal{P}$ with Lie algebra $\mathfrak{g}$. Partition functions are therefore specified with reference to $\mathcal{P}$ as they are computed via path-integrals over $G_\mathcal{P}$-bundles. Without specifying $\mathcal{P}$ the system is merely characterized by a "vector of partition functions" and the defect group of lines is data to this more general system. See [9] for further discussion.

Field theoretically line operators are of interest as order parameters to phase transitions, such as color confinement, and the area / perimeter law of the lines contained in $\mathcal{P}$ determine the phase structure of the gauge theory. See [7] for further discussion. They will later span representations to generalized global symmetries which are then of interest due to anomalies and selection rules.

1.1.2 Generalizations to d Dimensions

We continue with a slight generalization of the preceeding example and Yang-Mills theory with simply-laced gauge algebra $\mathfrak{g}$ of type ADE in arbitrary d spacetime dimensions. Electric dynamical particles and Wilson lines are treated analogously as in the previous example. Magnetic dynamical excitations and defects now generalize to codimension-3 excitations and probes. They remain labelled by elements of the previously introduced lattices. However, now the defect groups consists of electric lines and 't Hooft surfaces supported on submanifolds in dimension $d-3$. We introduce the defect group as

$$\mathbb{D}(\mathfrak{g}) = \mathbb{D}^{(1)}(\mathfrak{g}) \oplus \mathbb{D}^{(d-3)}(\mathfrak{g}) \cong \mathcal{Z}_{\text{elec}}(\mathfrak{g}) \oplus \mathcal{Z}_{\text{mag}}(\mathfrak{g})\,. \tag{1.5}$$

The pairing on the defect group is now

$$\langle\,\cdot\,,\cdot\,\rangle\;:\;\mathbb{D}^{(1)}(\mathfrak{g}) \times \mathbb{D}^{(D-3)}(\mathfrak{g}) \to \mathbb{Q}/\mathbb{Z}\,, \qquad \langle e,m\rangle = \exp\left(\frac{2\pi i}{|\mathcal{Z}(\mathfrak{g})|}em\right)\,. \tag{1.6}$$

No dyonic objects exist, but otherwise considerations run parallel. The change in dimension has as consequence that lines no longer braid with lines, similarly two 't Hooft surfaces do not braid.

Note that we can also add bosonic / fermionic matter. As long as the overall added dynamical states carry gauge charges valued in the root and coroot lattice the quotient computation is unchanged. In particular the above computations equally well apply to supersymmetric Yang-Mills theory as then all fields are in the adjoint representation.

1.2 Geometrization of Gauge Theory Defects

We now turn to our first example of an extra-dimensional construction. The M-theory setup we will consider engineers minimally supersymmetric 7D Yang-Mills theory and as such we will focus on exemplifying the derivation of (1.5) and (1.6) from the extra-dimensional engineering geometry. See [10, 11] for further discussion on relevant geometric engineering basics.

More concretely, we now consider M-theory on $\mathbb{R}^{1,6} \times X_4$ with ADE singularity $X_4 = \mathbb{C}^2/\Gamma_{\text{ADE}}$ and finite subgroup $\Gamma_{\text{ADE}} \subset SU(2)$. This background engineers 7D minimally supersymmetric Yang-Mills theory on $\mathbb{R}^{1,6}$ with Lie algebra $\mathfrak{g}$ which is the Lie algebra placed in correspondence with Γ_{ADE} via the McKay correspondence.

One way to see this is to consider the crepant resolution $\widetilde{X}_4$. The resolution deforms the space to be a smooth Calabi-Yau 2-fold at the cost of introducing a collection of exceptional spheres $\mathbb{P}^1_i$ or dual compactly supported 2-cocyles ω_i which pair as

$$\mathbb{P}^1_i \cdot \mathbb{P}^1_j = \frac{\omega_i \wedge \omega_j}{\text{vol}\,\widetilde{X}_4} = -\mathscr{C}_{ij}\,, \tag{1.7}$$

where $\mathscr{C}_{ij}$ is the Cartan matrix of $\mathfrak{g}$ and $\text{vol}\,\widetilde{X}_4$ is the volume form. See figure 1. When all scales of $\widetilde{X}_4$ are large then M-theory is well-approximated at low-energies by 11D supergravity and the effective 7D physics follows by reducing 11D supergravity on $\widetilde{X}_4$. This supergravity theory contains

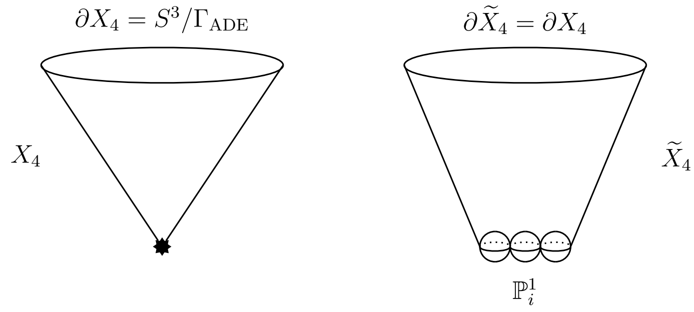


Figure 1: (i): Sketch of the ADE singularity $X_4 = \mathbb{C}^2/\Gamma_{\text{ADE}}$. (ii): Sketch of the ALE space $\widetilde{X}_4$ which is a crepant resolution of the ADE singularity $\widetilde{X}_4 \to X_4$.

a 3-form gauge field C and the 7D 1-form gauge fields A^i are then determined by the expansion

$$C = A^i \wedge \omega_i + \dots \,. \tag{1.8}$$

This leads in the 7D theory associated with $\widetilde{X}_4$ overall to the gauge algebra $\mathfrak{u}(1)^{\text{rank}\,\mathfrak{g}} = \mathfrak{h}$. There are also relevant massive particle states from M2-branes wrapped on the spheres $\mathbb{P}^1_j$. With respect to the i-th $U(1)$ factor they have charges $-\mathscr{C}_{ij}$. They are therefore associated with roots of $\mathfrak{g}$ and these particles should therefore be thought of as the W-bosons of the Higgsing

$$\mathfrak{g} \to \mathfrak{h} \,. \tag{1.9}$$

Conversely when all the spheres $\mathbb{P}^1_i$ are collapsed to zero volume these particles become massless and enhance the gauge symmetry from the Cartain subalgebra to the full non-Abelian gauge algebra.

The M2-branes can wrap arbitrary integral linear combinations of 2-spheres $\mathbb{P}^1_j$ and their gauge charges are arbitrary integral combinations of roots. This leads to the identification

$$H_2(\widetilde{X}_4; \mathbb{Z}) \cong \Lambda_{\text{root}}(\mathfrak{g}) \,. \tag{1.10}$$

An abstract level we have $\Lambda_{\text{weight}}(\mathfrak{g}) \cong \Lambda_{\text{root}}(\mathfrak{g})^*$, i.e., the weight lattice can be characterized as all rational linear combinations of roots that intersect integrally with all roots.[1] Parsing this in geometry one finds that compact 2-cycles are intersected similarly with all relative 2-cycles. We have the identification

$$H_2(\widetilde{X}_4, \partial\widetilde{X}_4; \mathbb{Z}) \cong \Lambda_{\text{weight}}(\mathfrak{g}) \,. \tag{1.11}$$

Here we have the asymptotic boundary $\partial\widetilde{X}_4 = S^3/\Gamma_{\text{ADE}}$ and the group of relative 2-cycles contains all compact and non-compact 2-cycles. Of course, as with the lattices there is a natural inclusion

$$H_2(X_4; \mathbb{Z}) \hookrightarrow H_2(X_4, \partial X_4; \mathbb{Z}) \,, \tag{1.12}$$

[1] See [12] for relevant basics in algebraic topology.

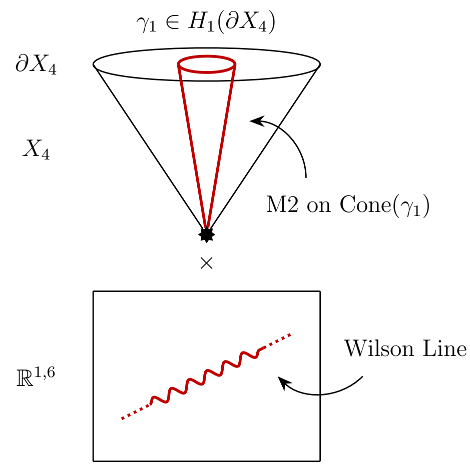


Figure 2: Sketch of the extra-dimensional construction of a Wilson line operator. An M2-brane wrapped on the non-compact 2-cycle Cone(γ_1) with $\gamma_1 \in H_1(\partial X_4)$ results due to Vol(Cone(γ_1)) $= \infty$ in an infinitely massive, and therefore non-dynamical, electric probe particle, i.e., a Wilson line.

and we therefore have the identity of quotients

$$\frac{H_2(X_4, \partial X_4; \mathbb{Z})}{H_2(X_4; \mathbb{Z})} \cong \frac{\Lambda_{\text{weight}}(\mathfrak{g})}{\Lambda_{\text{root}}(\mathfrak{g})} \,. \tag{1.13}$$

Up to this point we have simply geometrized the field theory discussion with no added insight. To proceed and leverage the extra-dimensional construction we next take the equality (1.13) at face value: the quotient of lattices setting the defect group is computed in geometry by removing all bulk features of the geometry X_4. Quotienting out by compact 2-cycles can here be viewed as setting these to zero indicating that $\Lambda_{\text{weight}}/\Lambda_{\text{root}}$ is an invariant under desingularization of the original ADE singularity. The ADE singularity is a cone and therefore fully determined by its link $\partial \widetilde{X}_4 = S^3/\Gamma_{\text{ADE}}$. The defect group therefore seems to be a function of S^3/Γ_{ADE} and the invariance of the above quotients is reflected in geometry by $\partial X_4 = \partial \widetilde{X}_4$.

This observation is made precise by noting that the lattices fit into the short exact sequence

$$0 \;\rightarrow\; \Lambda_{\text{weight}}(\mathfrak{g}) \;\rightarrow\; \Lambda_{\text{root}}(\mathfrak{g}) \;\rightarrow\; \mathcal{Z}(\mathfrak{g}) \;\rightarrow\; 0 \,, \tag{1.14}$$

and that this sequence fully geometrizes

$$0 \;\rightarrow\; H_2(\widetilde{X}_4, \partial \widetilde{X}_4; \mathbb{Z}) \;\rightarrow\; H_2(\widetilde{X}_4; \mathbb{Z}) \;\rightarrow\; H_1(\partial \widetilde{X}_4; \mathbb{Z}) \;\rightarrow\; 0 \,. \tag{1.15}$$

This sequence is an exact subsequence of the long exact sequence in relative homology of the pair $X_4, \partial X_4$. In particular we have

$$H_1(\partial \widetilde{X}_4; \mathbb{Z}) \cong H_1(\partial X_4; \mathbb{Z}) \cong \mathcal{Z}(\mathfrak{g}) \,. \tag{1.16}$$

The defect group $\mathbb{D}$ is therefore determined by the set of 1-cycles $\gamma_1 \in H_1(S^3/\Gamma_{\text{ADE}}; \mathbb{Z})$ contained in the asymptotic boundary of the engineering geometry.

Equation (1.16) also suggests an extra-dimensional construction for the Wilson lines $\mathbb{D}^{(1)}$ and 't Hooft surfaces $\mathbb{D}^{(4)}$ modulo screening effects. Consider the cone $\text{Cone}(\gamma_1) \in H_2(X_4, \partial X_4; \mathbb{Z})$ over $\gamma_1 \in H_1(S^3/\Gamma_{\text{ADE}}; \mathbb{Z})$. Wrap an M2-brane on $\text{Cone}(\gamma_1)$ resulting in a particle-like excitation in 7D. The mass of this particle is the product of the M2-brane tension with the volume $\text{Vol}(\text{Cone}(\gamma_1)) = \infty$ and therefore we have constructed an infinitely massive probe particle. This probe is a Wilson line. See figure 2. Analogously 't Hooft surfaces are constructed by wrapping M5-branes on the same classes $\text{Cone}(\gamma_1)$.

Another neat consequence of the geometrization is that the Dirac pairing

$$\langle \cdot, \cdot \rangle : \mathbb{D}^{(1)}(\mathfrak{g}) \times \mathbb{D}^{(4)}(\mathfrak{g}) \to \mathbb{Q}/\mathbb{Z}, \tag{1.17}$$

geometrizes to the linking pairing

$$\ell(\cdot, \cdot) : H_1(S^3/\Gamma_{\text{ADE}}) \times H_1(S^3/\Gamma_{\text{ADE}}), \tag{1.18}$$

which, using isomorphisms $\mathbb{D}^{(1)}(\mathfrak{g}) \cong H_1(S^3/\Gamma_{\text{ADE}})$ and $\mathbb{D}^{(4)}(\mathfrak{g}) \cong H_1(S^3/\Gamma_{\text{ADE}})$ which leave the type of wrapped brane implicit, can now be explicitly expressed as

$$\langle e, m \rangle = \exp\left(2\pi i\, \ell(e, m)\right) . \tag{1.19}$$

2 The Defect Group

The above discussion generalizes straightforwardly to a manifold of settings. See [13] for the original discussion in the context of 6D SCFTs and [14, 15] for other settings.

Let X be a non-compact geometry such that $\mathbb{R}^{1,d-1} \times X$ constitutes the geometric profile of some consistent background of IIA/IIB string theory or M-theory. We consider "purely geometric" backgrounds, e.g., at this point we exclude the possibility of brane sources. Then we engineer the non-gravitational theory

$$X \xrightarrow{\text{IIA/IIB/M}} \mathcal{T}_X . \tag{2.1}$$

If the internal space is a metric cone $X = \text{Cone}(\partial X)$ over a smooth link ∂X then the engineered theory is d-dimensional after taking the standard geometric engineering limits and we write $\mathcal{T}_X^{(d)}$. More general situations are possible.

IIA/IIB/M-theory come include diverse spectra of p-branes. The group of defects constructed via p-brane wrappings is then

$$\mathbb{D} = \bigoplus_m \mathbb{D}^{(m)}, \qquad \mathbb{D}^{(m)} \cong \bigoplus_{p\text{-branes}} \bigoplus_{p=k+m} \text{Tor}\, H_k(\partial X; \mathbb{Z})|_{\text{triv}} \tag{2.2}$$

where $\mathbb{D}^{(m)}$ is the group of defects with m-dimensional support in spacetime $\mathbb{R}^{1,d-1}$. We have

$$\text{Tor}\, H_k(\partial X; \mathbb{Z})|_{\text{triv}} = \text{Ker}\left(\text{Tor}\, H_k(\partial X; \mathbb{Z}) \to \text{Tor}\, H_k(X; \mathbb{Z})\right) . \tag{2.3}$$

which are precisely the cycles of the boundary which arise as the boundary of non-compact cycles in the bulk. The latter are the p-brane wrapping loci. With this p-branes with $p+1$ dimensional world volume are wrapped on $k+1$ dimensional non-compact cycles resulting in defects of dimension $p+1-(k+1)=p-k=m$ in spacetime.

Let us make some important observations which will guide much of the upcoming discussion and applications:

- Simplification: Screening Computation vs. Direct Geometric Construction. Instead of computing the full lattice of charges and their screening capabilities the extra-dimensional perspective allows for direct construction of the screened defect operators.
- Topological Structures: No requirement for holomorphy / supersymmetry. Traditionally, defect groups and related structures have been studied in holomorphic / supersymmetric compactifications which has simplified the understanding and verification of computations from field theory perspectives. However, the defect group is computed from topological structures.
- Geometric Engineering 101. The defect group is computed with respect to a fixed geometry, not a neighborhood in some moduli space or other deformation space. It is one of the simplest quantities associated directly to $\mathcal{T}_X$'s. The defect group may change across deformations of $\mathcal{T}_X$ and moduli spaces.
- Inherently non-Lagrangian and Relative. The defect group makes no reference to a Lagrangian and is particularly useful in studying inherently non-Lagrangian theories which are directly defined via X. The defect group does not require the existence of a polarization and is directly associated with the relative theory $\mathcal{T}_X$.
- Local Features. The defect group admits refinements to capture features of substrata of X. It directly captures how topological structures associated to various sets of degrees of freedom localized to singularities in X interact.
- Generalizations. In the above we restricted to purely geometric string backgrounds. However, "purely geometric" is not a physical concept. Both metric and non-metric supergravity fields may be interchanged by dualities and so the above constitutes a useful starting point for generalizations to non-geometric backgrounds.

3 Aspects of the Defect Group via Example

We now exemplify the observations of the preceding section.

3.1 Example: The Defect Group and RG Flow

In many gauge theories line defects realize order parameters for confinement. We now consider 4D $\mathcal{N}=1$ supersymmetric Yang-Mills theory, discuss the geometrization of its RG flow and compute the defect group as a function of energy scale and how it detects confinement. We begin recalling basics of the G_2-flop as laid out in [16].

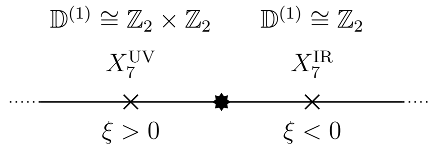


Figure 3: The confining transition of 4D $\mathcal{N} = 1$ supersymmetric Yang-Mills theory is geometrized by the G_2-flop. The UV geometry X_7^{UV} flops to the IR geometry X_7^{IR}. The defect group associated to both geometries changes indicating the area law of Wilson line operators and confinement of gauge degrees of freedom.

We take our starting point with M-theory on $\mathbb{R}^{1,3} \times X_7$ where $X_7 = \mathbb{S}(S^3)/\mathbb{Z}_2$ where $\mathbb{S}(S^3)$ is the total space of the spinor bundle over S^3. The space $\mathbb{S}(S^3)$ admits a G_2-holonomy metric, however, as a vector bundle the space is trivial and topologically $\mathbb{S}(S^3) = S^3 \times \mathbb{R}^4$. The $\mathbb{Z}_2$ acts via an isometry and consequently X_7 also admits a G_2-holonomy metric implying that minimal supersymmetry is preserved in 4D. The quotient is such that, as topological spaces, we have

$$X_7^{\text{UV}} = \mathbb{S}(S^3)/\mathbb{Z}_2 = S^3 \times (\mathbb{R}^4/\mathbb{Z}_2)\,. \tag{3.1}$$

The $\mathfrak{su}_2$ singularity $\mathbb{R}^4/\mathbb{Z}_2$ engineers a 7D supersymmetric Yang-Mills theory which is then suitably compactified on S^3 to 4D ultimately resulting in 4D $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with Lie algebra $\mathfrak{su}_2$. The 4D gauge coupling and 7D gauge coupling relate as

$$\frac{1}{g^2_{\text{YM,4D}}} = \frac{\text{Vol}(S^3)}{g^2_{\text{YM,7D}}}\,. \tag{3.2}$$

Under RG flow 4D $\mathcal{N} = 1$ supersymmetric Yang-Mills theory is driven to strong coupling and then confines. The increasing 4D gauge coupling is geometrized as a decrease of $\text{Vol}(S^3)$. Slightly shifting our perspective we can think of X_7^{UV} as the $\mathbb{Z}_2$ quotient of the hypersurface

$$\xi = u_1^2 + u_2^2 + u_3^2 + u_4^2 - (v_1^2 + v_2^2 + v_3^2 + v_4^2)\,. \tag{3.3}$$

where $\xi^{3/2} \propto \text{Vol}(S^3)$ and the $\mathbb{Z}_2$ quotient acts on the coordinates v_i. The confining transition is realized by starting out with $\xi > 0$ and then flowing to $\xi < 0$. This is referred to as the G_2-flop. The flopped geometry with $\xi < 0$ is

$$X_7^{\text{IR}} = \mathbb{S}(S^3)/\mathbb{Z}_2 = S^3/\mathbb{Z}_2 \times \mathbb{R}^4\,, \tag{3.4}$$

which is smooth, i.e., the singularities responsible for the gauge theory degrees of freedom have disappeared as expected by confinement.

Domain walls are realized by an M5-brane on $S^3/\mathbb{Z}_2$. Confining strings are realized by M2-branes wrapped on the Hopf circle of $S^3/\mathbb{Z}_2$ [17]. Note further that $\partial X_7^{\text{UV}} = \partial X_7^{\text{IR}}$ which is another indication that we are considering a phase transition within a given theory which is determined by fixed extra-dimensional boundary conditions.

We can now compute the defect group across the flop, we have

$$\begin{aligned} H_1(\partial X_7^{\rm UV})|_{\rm triv} &\cong H_1(S^3/\mathbb{Z}_2)|_{\rm triv} \cong \mathbb{Z}_2\,, \\ H_1(\partial X_7^{\rm IR})|_{\rm triv} &\cong H_1(S^3/\mathbb{Z}_2)|_{\rm triv} \cong 0\,, \\ H_4(\partial X_7^{\rm UV})|_{\rm triv} &\cong \left(H_3(S^3)\otimes H_1(S^3/\mathbb{Z}_2)\right)|_{\rm triv} \cong \mathbb{Z}_2\,, \\ H_4(\partial X_7^{\rm IR})|_{\rm triv} &\cong \left(H_3(S^3)\otimes H_1(S^3/\mathbb{Z}_2)\right)|_{\rm triv} \cong \mathbb{Z}_2\,, \end{aligned} \tag{3.5}$$

and therefore:

$$\mathbb{D}^{(1)}_{\rm UV,0} \cong \mathbb{Z}_2^{(W)}\times \mathbb{Z}_2^{(H)}\,, \qquad \mathbb{D}^{(1)}_{\rm IR,0}\cong \mathbb{Z}_2^{(H)}\,. \tag{3.6}$$

In particular the non-compact 2-cycle allowing for the construction of the Wilson line via an M2-brane wrapping is lost upon flopping. Here the subscript zero refers to the background being purely geometric, see below.

The change in the defect group can be traced in greater detail. For this, attempt to build a Wilson line in $X_7^{\rm IR}$ starting from an asymptotic $\gamma_1\in H_1(\partial X_7^{\rm IR};\mathbb{Z})$. In the UV geometry, which has the same boundary, such a 1-cycle can be contracted to a point in the bulk, constructing a non-compact 2-cycle. In contrast, in the IR geometry γ_1 is not contractible, extending it into the bulk results in a cylinder which ends on the Hopf circle of the zero section $S^3/\mathbb{Z}_2$. So far this cylinder projects onto a line in spacetime - the would-be Wilson line. However, this is not a consistent configuration, to obtain a closed wrapping locus of the M2-brane we must cap-off the cylinder. This is achieved by an M2-brane wrapped on a disk in spacetime with boundary on the would-be Wilson line, and internally wrapping the Hopf circle. This configuration corresponds to the confining string giving area-law to the Wilson line.

There is a similar story where the UV background is altered by turning on the period $\int_{S^3} C/2\pi = 1$. In the 4D gauge theory this amounts to shifting the theta-angle as $\theta\to\theta+2\pi$ and as such this background still supports the same 4D gauge theory along its ADE singularities. In this setting we compute

$$\mathbb{D}^{(1)}_{\rm UV,2\pi} \cong \mathbb{Z}_2^{(W)}\times \mathbb{Z}_2^{(H)}\,, \qquad \mathbb{D}^{(1)}_{\rm IR,2\pi}\cong \mathbb{Z}_2^{(W+H)}\,, \tag{3.7}$$

which corresponds to flopping into to the second vacua of the 4D $\mathfrak{su}(2)$ $\mathcal{N}=1$ Yang-Mills theory.

3.2 Example: Topology and Non-SUSY Backgrounds

The defect group computes directly from topological structures of the engineering geometry. Given some unkown $\mathcal{T}_X$ the defect group therefore serves as a first piton in determining properties of $\mathcal{T}_X$ which in turn constrain its non-topological features. We demonstrate this approach by considering non-supersymmetric string backgrounds and highlighting again how the geometry allows us to avoid a much more complicated screening construction. We begin reviewing basics of the relevant setups focussing on [18, 19].

We take our starting point with IIA string theory on $X_6=\mathbb{R}^6/\Gamma$ with Γ is a finite subgroup of $SO(6)$. The internal space X_6 is a Calabi-Yau 3-fold only iff $\Gamma\subset SU(3)$ and in general this background preserves no supersymmetry in 4D. Even worse, it has a closed string tachyon in some twisted sector. However, as far as tachyons go this instability is rather mild and ultimately leads to a rolling solution which is fully characterized by X_6 being time-dependent. Said differently we are considering IIA string theory $\mathbb{R}^3 \times X_7$ where X_7 is fibered over the spacetime time coordinate and we prepare the system such that at some initial time $t_0 \in \mathbb{R}_t$ the fiber is X_6.

In any case, it turns out that at every instance of time we can reliable compute the defect group of this system. Before discussing the geometric computation let us discuss the screening approach and compute the relevant charge lattices and Dirac pairings. The effective physics propagates in 4D, we are therefore concerned with line operators and our discussion will parallel the previously discussed 4D Yang-Mills example.

First, we discuss the charge lattice. For this we consider a D0-brane probe of X_6. At the singularity this brane fractionates into D0/D2/D4-branes and the resulting quantum mechanics therefore knows about the full set of charges carried by states constructed from these objects [20] (see also [21]). This quantum mechanics is a quiver quantum mechanics. It is characterized by a bosonic and fermionic quiver Q^B, Q^F with adjacency matrices a^B_{ij}, a^F_{ij} where $i, j, = 1, \dots, R$ with $R = \mathrm{rank}(Q^B) = \mathrm{rank}(Q^F)$. The quivers Q^B, Q^F are linear McKay quivers associated to the reference representation **3** and **4** respectively. Taking the free group over the quiver nodes we obtain the lattice

$$\Lambda \cong \mathbb{Z}^R \,. \tag{3.8}$$

Next, we turn to the Dirac pairing. The adjacency matrices give the linear maps

$$a^B_{ij}, a^F_{ij} \,:\, \mathbb{Z}^R \to \mathbb{Z}^R \,. \tag{3.9}$$

It turns out that the Dirac pairing is set by the antisymmetrization $\Omega_{ij} = a^F_{ij} - a^F_{ji}$ [22]. The dynamical charges lie in the image of Ω, the defect group of lines therefore computes to

$$\mathrm{Tor}\,\mathrm{Coker}\,\Omega \cong \mathbb{D}^{(1)} \,. \tag{3.10}$$

With $\partial X_6 = S^5/\Gamma$ this computation is matched in geometry as follows

$$\mathbb{D}^{(1)} \cong \mathrm{Tor}\, H_1^{\mathrm{orb}}(S^5/\Gamma) \oplus \left[\mathrm{Tor}\, H_1^{\mathrm{orb}}(S^5/\Gamma)\right]^\vee , \tag{3.11}$$

Here $[\mathrm{Tor}\, H_1^{\mathrm{orb}}(S^5/\Gamma)]^\vee$ is the subset of $\mathrm{Tor}\, H_3^{\mathrm{orb}}(S^5/\Gamma)$ which links non-trivially with $\mathrm{Tor}\, H_1^{\mathrm{orb}}(S^5/\Gamma)$. In general there are more 3-cycles than 1-cycles indicating a 2-group structure as we will explain elsewhere. Here $\vee$ indicates the Pontryagin dual to a finite group H, i.e., $H^\vee = \mathrm{Hom}(H, U(1))$.

We again note the incredible simplification geometrization entails. The cokernal torsion subgroup $\mathrm{Tor}\,\mathrm{Coker}\,\Omega$ is a fermionic bulk quantity deriving from open string degrees of freedom. The groups $H_k^{\mathrm{orb}}(S^5/\Gamma)$ are bosonic boundary quantities deriving from closed string degrees of freedom. The homology theory H_*^{orb} derives from Chen-Ruan orbifold cohomology [23, 24].

This result geometrizes in a much stronger sense away from the torsion subgroup. With $\partial X_6 = S^5/\Gamma$, where Γ can have arbitrary fixed point structure on S^5, we have

$$\mathrm{Coker}\,\Omega \cong H^0_{\mathrm{orb}}(\partial X; \mathbb{Z}) \oplus H^2_{\mathrm{orb}}(\partial X; \mathbb{Z}) \oplus H^4_{\mathrm{orb}}(\partial X; \mathbb{Z}) = K^0(\partial X; \mathbb{Z}) \,. \tag{3.12}$$

Here, the defect group $\mathbb{D}^{(1)}$ is isomorphic to the torsional subgroup of the K-theory group $K^0(\partial X; \mathbb{Z})$. This is completely consistent with our constructions of defects via wrapped D-branes as D-brane charges take values in K-theory. See [25] for K-theory basics.

3.3 Example: Non-Lagrangian and Relative Theories

The defect group $\mathbb{D}$ is computed from the geometry X. Boundary conditions of relevant supergravity fields at ∂X, which are required to specify an absolute version of the relative theory $\mathcal{T}_X$ do not enter $\mathbb{D}$. The defect group is therefore attached directly to relative theories in non-Lagrangian fashion. To exemplify this aspect we now discuss the famous 6D $\mathcal{N} = (2,0)$ SCFTs.

We take our starting point with IIB string theory on $\mathbb{R}^{1,5} \times X_4$ where $X_4 = \mathbb{C}^2/\Gamma_{\text{ADE}}$ with finite subgroup $\Gamma_{\text{ADE}} \subset SU(2)$ corresponding via the McKay correspondence to the simply laced Lie algebra $\mathfrak{g}$. This background realizes the famous 6D $\mathcal{N} = (2,0)$ SCFT labelled by the Lie algebra $\mathfrak{g}$. This SCFT is a theory of mutually non-local tensionless strings and non-Lagrangian. The strings can be identified as D3-branes wrapped on the vanishing cycles of X_4. Denote by $\widetilde{X}_4$ the ALE space realizing a crepant resolution $\widetilde{X}_4 \to X_4$. The crepant resolution geometrizes the deformation of the SCFT to a point on its tensor branch. Under such deformations dynamical strings pick up tension and the effective description includes Abelian gauge factors. One can then introduce the lattice of charges for string-like excitations

$$\Lambda_{\text{string}} \cong H_2(\widetilde{X}_4; \mathbb{Z}) \,, \tag{3.13}$$

which is geometrized by the internal wrapping locus of the D3-branes in $\widetilde{X}_4$. The dual lattice with respect to the intersection pairing is denoted $\Lambda^*_{\text{string}}$. Unimodular lattices satisfy $\Lambda_{\text{string}} = \Lambda^*_{\text{string}}$. Unimodularity implies that the 6D model has a well-defined partition function. For non-unimodular lattices the theory generically only posses a vector of partition functions and is intrinsically relative. Nonetheless we can compute the group of 2-surface defect operators

$$\mathbb{D}^{(2)} \cong \Lambda^*_{\text{string}}/\Lambda_{\text{string}} \cong \text{Tor}\, H_1(\partial X_4; \mathbb{Z}) \cong \text{Ab}(\Gamma_{\text{ADE}}) \tag{3.14}$$

where $\text{Ab}(\cdot)$ denotes the abelianization of groups. See [13] for further details.

3.4 Example: 2-group Structures

Often interesting interacting systems are constructed in string theory via stratified singularities. In the simplest case, there is a generic singular locus which enhances along some sublocus to a higher-codimension singularity. Various sets of localized degrees of freedom populate distinct singular strata and to each we can associate a defect group and study their interplay. In the simplest case this results in 2-group symmetries. To exemplify this interplay we now discuss certain 5D SCFTs engineered by non-isolated singularities in M-theory. See [26] for the original discussion.

We take our starting point with M-theory on $\mathbb{R}^{1,5} \times X_6$ with Calabi-Yau 3-fold $X_6 = \mathbb{C}^3/\mathbb{Z}_{2n}$ where the $\mathbb{Z}_{2n}$ acts via phase rotation by some $2n$-th root of unity on the three complex coordinates with weights $(1, 1, -2)$. Along $z_1, z_2 = 0$ and $z_3 \neq 0$ this geometry has normal geometry $\mathbb{C}^2/\mathbb{Z}_2$ and therefore $\mathbb{R}^{1,5} \times (\mathbb{C}/\mathbb{Z}_{2n} \setminus \{0\})$ supports a 7D $\mathfrak{su}(2)$ supersymmetric Yang-Mills theory. Here $\mathbb{C}$ is parametrized by z_3 and acted on with weight -2 and therefore the faithful quotient actually is $\mathbb{C}/\mathbb{Z}_{2n} = \mathbb{C}/\mathbb{Z}_n$. At the origin of $\mathbb{C}/\mathbb{Z}_n$ the codimension-4 singularity enhances to a codimension-6

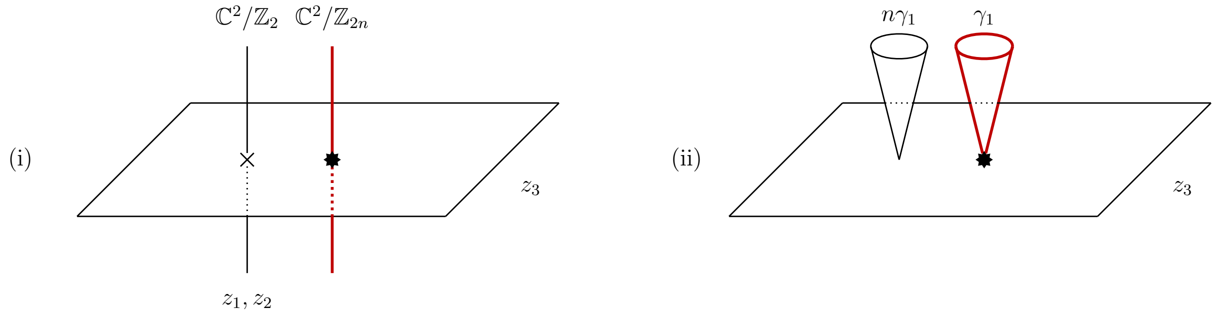


Figure 4: In (i) we sketch the geometry $\mathbb{C}^3/\mathbb{Z}_{2n}$. Along $z_1, z_2 = 0$ parametrized by $z_3 \neq 0$ there is an $\mathfrak{su}_2$ singularity. At the origin the normal geometry enhances to $\mathbb{C}^2/\mathbb{Z}_{2n}$. In (ii) we sketch a non-compact 2-cycle of the normal geometry $\mathbb{C}^2/\mathbb{Z}_{2n}$ as a cone over γ_1. Upon taking n copies of this 2-cycle it can be deformed into the generic normal geometry $\mathbb{C}^2/\mathbb{Z}_2$.

singularity. The codimension-6 singularity supports the 5D SCFT degrees of freedom. See subfigure (i) of figure 4 for a sketch of the geometry.

Let us focus on the defect group of lines $\mathbb{D}^{(1)}$ which are again constructed by wrapping M2-branes on non-compact 2-cycles. There are two sets of non-compact 2-cycles to consider. Denoting the generic normal geometry as $N_g = \mathbb{C}^2/\mathbb{Z}_2$ and the exceptional normal geometry as $N_e = \mathbb{C}^2/\mathbb{Z}_{2n}$. We can then consider wrapping M2-branes on the classes

$$H_2(N_g, \partial N_g; \mathbb{Z}) \cong \mathbb{Z}_2\,, \qquad H_2(N_e, \partial N_e; \mathbb{Z}) \cong \mathbb{Z}_{2n}\,. \tag{3.15}$$

The former construct Wilson lines of the 7D supersymmetric Yang-Mills theory, we have $\mathbb{D}^{(1)}_{7\text{D}} \cong \mathbb{Z}_2$. The latter construct electric line operators of the 5D SCFT. However, note that n copies of the a generator of $H_2(N_e, \partial N_e; \mathbb{Z})$ can be deformed off the exceptional normal geometry N_e into the generic normal geometry N_g and therefore $\mathbb{D}^{(1)}_{5\text{D}} \cong \mathbb{Z}_{2n}/\mathbb{Z}_2 \cong \mathbb{Z}_n$. All line operators fit into the short exact sequence

$$0 \;\to\; \mathbb{D}^{(1)}_{7\text{D}} \;\to\; \mathbb{Z}_{2n} \;\to\; \mathbb{D}^{(1)}_{5\text{D}} \;\to\; 0\,, \tag{3.16}$$

which does not split whenever n is even.

After taking a geometric engineering limit which reduces the 7D supersymmetric Yang-Mills locus to a flavor brane, such that $\mathcal{T}_X$ is an honest 5D theory, we find a 2-group like structure: n copies of the 5D SCFT line operators are equivalent to a flavor Wilson line.

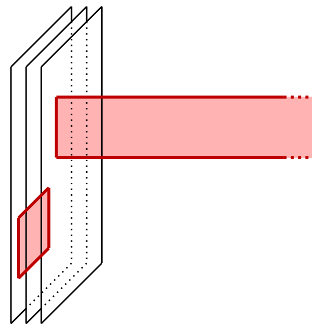


Figure 5: Sketch of a semi-infinite M2-brane with one end on a stack of M5-branes. The tensionless strings on the M5-brane worldvolume originate from M2-branes with both end on M5-branes.

3.5 Example: Generalizations to Non-Geometric Duality Frames

We now present a generalization to systems $\mathcal{T}_X$ where interacting degrees of freedom are localized along membranes. One large class of such theories are realized by M5-branes wrapped on a Riemann surface $\Sigma_{g,n}$ of genus g with n punctures. The resulting theories are 4D $\mathcal{N}=2$ theories and may be either Lagrangian or non-Lagrangian. See [27] for further details on this construction.

We take our starting point with noting that the 6D $\mathcal{N}=(2,0)$ SCFTs of type $\mathfrak{su}(N)$ can be engineered both in M-theory using stacks of N M5-branes or in IIB via the ordifold singularity $\mathbb{C}^2/\mathbb{Z}_N$. In the former, we will neglect center of mass modes as these will have little consequence for the below discussion for reasons which will become clear in later lectures.

Surface defects are constructed in the latter from D3-branes wrapped on non-compact 2-cycles of $\mathbb{C}^2/\mathbb{Z}_N$. Following these through the duality chain we learn unsurprisingly that in M-theory surface defects are constructed from semi-infinite M2-branes, ending on the M5-branes and stretching to the asymptotic S^4 boundary of the extra-dimensional space $\mathbb{R}^5$. See figure 5.

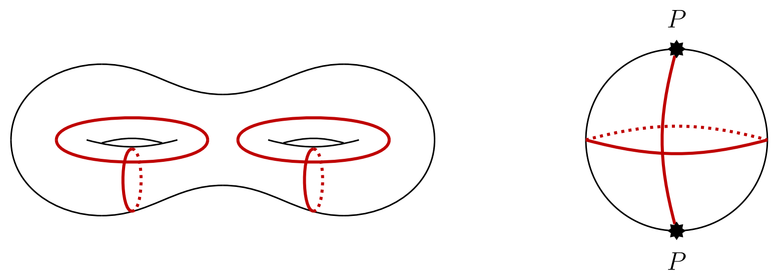
Wrapping this configuration on a Riemann surface $\Sigma_{g,0}$ of genus g and with no punctures these semi-infinite M2-branes are further distinguished by which of the A- or B-cycles of the Riemann surface they wrap. We therefore have that in going from 6D to 4D by compactifying on $\Sigma_{g,0}$ the defect group $\mathbb{D}^{(2)}\cong\mathbb{Z}_N$ of 6D surface defects contributes

$$\mathbb{D}^{(1)}\cong\mathbb{Z}_N^{2g}\,, \tag{3.17}$$

to the 4D defect group of lines. See subfigure (i) of figure 6. See [28] for more details and examples.

Consider next the case where $\Sigma_{0,n}$ has n identical punctures.[2] The contributions to the defect group of lines from punctures is determined via the spectral curve $\widetilde{\Sigma}_{0,n}\subset T^*\Sigma_{0,n}$ of the system.

[2] For example, 4D $\mathcal{N}=2$ Yang-Mills theory may be constructed via $\Sigma_{0,2}$ (the twice punctured sphere) where both punctures are of so-called irregular P_0-type. Subfigure (ii) of figure 6 then shows the extra-dimensional cycles associated with Wilson and 't Hooft lines.



(i) (ii)

Figure 6: (i): We sketch an unpunctured genus g Riemann surface ($g = 2$) and indicate the A- and B-cycles associated with line defects. We sketch a sphere with 2 punctures P at its north and south pole and indicate non-trivial 1-cycles introduced by the puncturing.

The spectral curve is an $N : 1$ covering of the punctured Riemann surface

$$\widetilde{\Sigma}_{0,n} \xrightarrow{N:1} \Sigma_{0,n} \,. \tag{3.18}$$

This covering can be described by N local holomorphic 1-forms $\lambda_i = \lambda_i(z)dz$ on $\Sigma_{0,n}$. They are the eigenvalues of an $\mathfrak{su}(N)$ valued Higgs field and correspond to the weights of $\mathfrak{g} = \mathfrak{su}(N)$. The linear combinations $\alpha_k = \lambda_k - \lambda_{k+1}$ with $k = 1, \dots, N-1$ correspond to the roots of $\mathfrak{su}(N)$. More precisely, at each point z of the Riemann surface $\Sigma_{0,n}$ the set of $\alpha_i(z)$ can be viewed as spanning the root lattice $\Lambda_{\text{root}}(z) \cong \Lambda_{\text{root}}(\mathfrak{g})$. All of these 1-forms are not globally defined and may permute along non-trivial paths on $\Sigma_{0,n}$. As we have specialized to $g = 0$, i.e., the sphere, such non-trivial paths are curves linking one or more punctures. Consider a path linking one of the punctures, then we can associate to the puncture P the monodromy on the root system

$$M_P \,:\, \Lambda_{\text{root}}(\mathfrak{g}) \;\to\; \Lambda_{\text{root}}(\mathfrak{g})\,, \tag{3.19}$$

that is, following the root $\alpha_i(z_*) \in \Lambda_{\text{root}}(z_*) \cong \Lambda_{\text{root}}(\mathfrak{g})$ along a small circle centered on a puncture, and starting and ending at z_* it is found to return to the combination $(M_P)_{ij}\alpha_j(z_*) \in \Lambda_{\text{root}}(z_*) \cong \Lambda_{\text{root}}(\mathfrak{g})$. This monodromy is independent of z_*.

Regular punctures have a simple pole and are therefore characterized by trivial monodromies $M = \text{Id}$. Irregular punctures P may have non-trivial monodromies and then we compute

$$\text{Tor}(\text{Coker}(M_P - \text{Id})) \cong K_P \oplus L_P^2\,, \tag{3.20}$$

for some finite groups K_P, L_P. This split in contributions comes down to the two ways in which punctures can add line defects to the system. First, associated with K_P, they can provide new 1-cycles on $\Sigma_{0,n}$ which are characterized by starting and ending at punctures. See subfigure (ii) of figure 6. Second, associated with L_P^2, we note that punctures should be viewed as codimension-2 defects on $\Sigma_{0,n}$ which are spacetime filling. They are defect QFTs and can add degrees of freedom, including line operators. The group L_P^2 is associated with pairs of electric and magnetic line defects localized to the puncture P. Putting these two contributions together the defect group of lines associated to $\Sigma_{0,n}$ with n identical punctures P computes to

$$\mathbb{D}^{(1)} \cong K_P^{2n-2} \oplus \left(\bigoplus_P L_P^2 \right) . \tag{3.21}$$

See [29] for further details and generalizations to $\Sigma_{g,n}$ with collections of non-identical punctures.

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