Lecture 2: Symmetry Operators in Geometric Engineering and Holography
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Abstract: We discuss extra-dimensional constructions of symmetry operators and their utility.

1 Introduction and Motivating Examples

In the last lecture we introduce the defect group. We focussed on non-gravitational systems with extra-dimensional constructions in string theory and utilized the engineering geometry X in computing and describing the systems defect group. The main insight was that elements of the defect group admitted construction starting from cycles in the asymptotic boundary ∂X and the relevant spectrum of p-branes.

In this lecture we discuss a completely analogous construction for the symmetry operators of non-gravitational systems with extra-dimensional constructions in string theory. Further, these will naturally act on the defect group, which, in the simplest cases, furnish representations of these symmetries.

1.1 Symmetry Operators in Gauge Theories

We begin with a field theoretic discussion of topological symmetry operators in gauge theories preparing first extra-dimensional analyses which will map onto the introduced structures. We take our starting point with Yang-Mills gauge theory.

1.1.1 Topological Operators in Yang-Mills theory

Consider Yang-Mills theory in d dimensions with simply-laced gauge algebra $\mathfrak{g}$ of type ADE in Cartan's classification and action

$$S[\mathfrak{g}] = -\frac{1}{4g_{\text{YM}}^2}\int \text{Tr}(F\wedge *F)\,, \tag{1.1}$$

with field strength $F = dA + A\wedge A$ and connection 1-form A valued in the adjoint representation of the Lie algebra $\mathfrak{g}$. Similar to how Wilson lines may be interpreted as electric probe particles we can introduce electric probe vortices. These are codimension-2 objects and characterized by the holonomy of the gauge field A along closed linking paths. For example, in 4D such an electric vortex is supported on a surface. To characterize such vortices in greater detail assume the gauge group is G, then holonomies are valued in G modulo gauge transformations and are therefore labelled by conjugacy classes $[g] \in \text{Conj}(G)$. Such electric operators $U_{[g]}$ are called Gukov-Witten operators and they are topological whenever g is an element of the center $Z_G \subset G$.

Complementarily, we have topological magnetic operators supported on surfaces characterized by classes $w_2(A) \in H^2(X, \pi_1(G))$. These measure the magnetic flux of a codimension-3 't Hooft

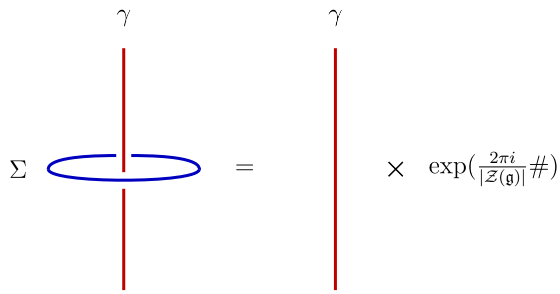


Figure 1: Sketch of a line operator linking a topological Gukov-Witten operator and the very same line operator in isolation. The two configurations differ by the Aharanov-Bohm phase picked up by the probe particle along the path γ linking the vortex operator on Σ.

operator modulo screening effects and can be understood nicely via an obstruction theoretic perspective [1]. We have the subgroup $\pi_1(G) \subset \mathcal{Z}(\mathfrak{g})$ and therefore considering all global forms to the ADE Lie algebra $\mathfrak{g}$ we see that the topological operators across all polarizations are drawn from the group

$$\mathcal{Z}_{\text{elec}}(\mathfrak{g}) \oplus \mathcal{Z}_{\text{mag}}(\mathfrak{g})\,, \tag{1.2}$$

where $\mathcal{Z}_{\text{elec}}(\mathfrak{g}) \cong \mathcal{Z}_{\text{mag}}(\mathfrak{g}) \cong \mathcal{Z}(\mathfrak{g})$ where $\mathcal{Z}(\mathfrak{g})$ is the center of the simply connected Lie group with Lie algebra $\mathfrak{g}$. We will therefore denote topological Gukov-Witten operators as U_z and topological magnetic flux operators as V_z with $z \in \mathcal{Z}(\mathfrak{g})$.

Topological Gukov-Witten operators naturally act on Wilson line operators. Consider a Wilson line linking a Gukov-Witten operator, viewing it as an infinitely heavy probe particle dragged around Gukov-Witten operator we find the overall configuration to be rotated by a phase compared to the line operator inserted in the absence of a Gukov-Witten operator. In equations we have

$$\langle U_z(\Sigma) W_{\mathcal{R}}(\gamma) \dots \rangle = \text{Hol}(\gamma, \Sigma, \mathcal{R})^z \langle W_{\mathcal{R}}(\gamma) \dots \rangle = \exp\left(\frac{2\pi i z N_{\mathcal{R}}}{|\mathcal{Z}(\mathfrak{g})|} \text{Link}(\Sigma, \gamma) \right) \langle W_{\mathcal{R}}(\gamma) \dots \rangle\,, \tag{1.3}$$

where $\text{Link}(\Sigma, \gamma)$ is the Gaussian linking number and $N_{\mathcal{R}}$ is the N-ality of the representation $\mathcal{R}$. Here, we assumed $\mathcal{Z}(\mathfrak{g})$ to be cyclic and identified $z \in \mathcal{Z}(\mathfrak{g})$ with an integer $z = 0, \dots, |\mathcal{Z}(\mathfrak{g})| - 1$.

We remark that (1.3) is largely agnostic to the representation $\mathcal{R}$, only its N-ality enters. The N-ality of a Wilson line $W_{\mathcal{R}}$ was precisely its label in the defect group $N_{\mathcal{R}} \in \mathbb{D}^{(1)}$ and for this reason (1.3) will factor through the screening and allow us to discuss the action of topological Gukov-Witten operators and magnetic flux operators on the much simpler group $\mathbb{D}$ associated with non-dynamical probes modulo screening.

The operators $U_z(\Sigma_{d-2}), V_z(\Sigma_2)$ fuse according to the group law of the center subgroup $\mathcal{Z}(\mathfrak{g})$:

$$U_z(\Sigma_{d-2}) \otimes U_{z'}(\Sigma_{d-2}) = U_{z+z'}(\Sigma_{d-2})\,, \qquad V_z(\Sigma_2) \otimes V_{z'}(\Sigma_2) = V_{z+z'}(\Sigma_2)\,. \tag{1.4}$$

They generate an abelian invertible 1-form and $(d-3)$-form symmetries [2].

When discussing the defect group we found maximal mutually local subsets $\mathcal{P} \subset \mathbb{D}$ to specify the gauge group $G_{\mathcal{P}}$ with Lie algebra $\mathfrak{g}$. Such choices of $\mathcal{P}$ also specify which of the operators U, V

are non-trivial in the $G_{\mathcal{P}}$-gauge theory. These are simply the U, V's which act non-trivially via (1.4) on lines and surfaces in $\mathcal{P}$. As a consequence, for example in 4D, the 1-form symmetry group is then simply $\mathcal{P}^\vee$. Here $\vee$ denotes Pontryagin duality which given a group H produces the dual group $H^\vee = \text{Hom}(G, U(1))$ and where for abelian finite groups we have $H^\vee \cong H$ non-canonically.

1.2 Geometrization of Gauge Theory Symmetry Operator

We now return to the M-theory engineering minimally supersymmetric 7D Yang-Mills theory and as such we will focus on giving the extra dimensional construction of the operators U, V generating 1-form, 4-form symmetries respectively.

Concretely, consider M-theory on $\mathbb{R}^{1,6} \times X_4$ with ADE singularity $X_4 = \mathbb{C}^2/\Gamma_{\text{ADE}}$ and finite subgroup $\Gamma_{\text{ADE}} \subset SU(2)$. This background engineers 7D minimally supersymmetric Yang-Mills theory on $\mathbb{R}^{1,6}$ with Lie algebra $\mathfrak{g}$ which is the Lie algebra placed in correspondence with Γ_{ADE} via the McKay correspondence (see Lecture 1 for further details, and the references given there).

Recall that Wilson lines $\mathbb{D}^{(1)}$ and 't Hooft surfaces $\mathbb{D}^{(4)}$ are constructed from M2-branes and M5-branes wrapped on non-compact 2-cycles of X_4 respectively. The symmetry topological field theory perspective we present later heavily suggests that, for discrete symmetries, defect and symmetry operators should be constructed from the same objects in string theory. This is because there, defect and symmetry operators can simply be rotated into another and their distinction boils down to orientation in the auxiliary slab and certain boundary conditions.

With this in mind we aim to construct topological operators U, V supported in dimensions 5 and 2 from M5- and M2-branes respectively. However, note immediately that any membrane wrapped in the bulk of X_4 cannot be topological. Counting dimensions, note next that the extra-dimensional support of both wrapping loci must be 1-dimensional. The asymptotic boundary $\partial X_4 = S^3/\Gamma_{\text{ADE}}$ has one-cycles

$$\gamma_1 \in H_1(S^3/\Gamma_{\text{ADE}}) \cong \mathcal{Z}(\mathfrak{g})\,. \tag{1.5}$$

To construct the topological Gukov-Witten operators and magnetic flux operators we will "wrap" the M5- and M2-branes on such asymptotic 1-cycles respectively. Let us now give meaning to this notion of wrapping and check it for consistency. See figure 2.

First, any non-topological interaction between the 7D gauge theory degrees of freedom localized to the singularity at the tip of the cone X_4 and those of M5- and M2-brane worldvolumes scales with the distance between these. Therefore, we find this asymptotic wrapping to produce an object coupled at most topologically to the 7D supersymmetric Yang-Mills theory. Second, non-topological excitations on the asymptotically wrapped branes are suppressed by

$$\text{Vol}\,\gamma_1 = \infty\,. \tag{1.6}$$

We later return to this condition in the setting of D-branes where this condition explicitly localizes the world volume degrees of freedom to configurations of vanishing action. Overall, such asymptotic brane wrappings therefore result in spacetime subloci supporting only topological degrees of freedom

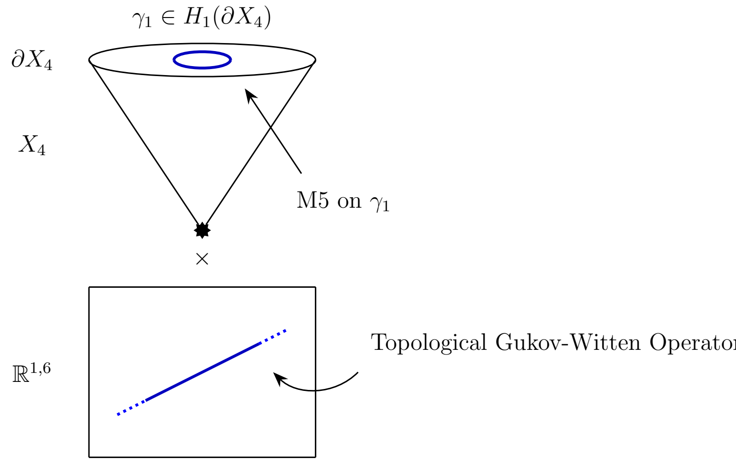


Figure 2: Sketch of the extra-dimensional construction of a topological operator acting on Wilson line operator. An M5-brane wrapped on the asymptotic 1-cycle $\gamma_1 \in H_1(\partial X_4)$ results due to $\mathrm{Vol}(\gamma_1) = \infty$ in a topological operator.

and we therefore schematically associate to these the operators

$$\begin{aligned} \mathcal{U}(\Sigma_5) &= \int [D\Phi] \exp\left(2\pi i \int_{\Sigma_5 \times \gamma_1} \mathcal{L}^{\text{top}}_{\text{M5}}(\Phi)\right) , \\ \mathcal{V}(\Sigma_2) &= \int [D\Phi] \exp\left(2\pi i \int_{\Sigma_2 \times \gamma_1} \mathcal{L}^{\text{top}}_{\text{M2}}(\Phi)\right) , \end{aligned} \tag{1.7}$$

in obvious notation. To make contact with the gauge theory operators U, V we only need to recall their couplings to 11D supergravity gauge fields as

$$\mathcal{L}^{\text{top}}_{\text{M5}} = C_6 + \dots , \quad \mathcal{L}^{\text{top}}_{\text{M2}} = C_3 + \dots . \tag{1.8}$$

In this example the abbreviated terms can be neglected and the path-integral for $\mathcal{U}, \mathcal{V}$ trivializes. Overall we therefore find[1]

$$\begin{aligned} U_z(\Sigma_5) &= \exp\left(2\pi i \int_{\Sigma_5 \times z\gamma_1} C_6\right) = \exp\left(\frac{2\pi i z}{|\mathcal{Z}(\mathfrak{g})|} \int_{\Sigma_5 \times \Gamma_2} G_7\right) , \\ V_z(\Sigma_2) &= \exp\left(2\pi i \int_{\Sigma_2 \times z\gamma_1} C_3\right) = \exp\left(\frac{2\pi i z}{|\mathcal{Z}(\mathfrak{g})|} \int_{\Sigma_2 \times \Gamma_2} G_4\right) , \end{aligned} \tag{1.9}$$

where used the torsion relation $\partial \Gamma_2 = |\mathcal{Z}(\mathfrak{g})| \gamma_1$ and $G_4 = dC_3$ and $G_7 = dC_6$. It now follows directly that Wilson lines and ‘t Hooft surfaces constructed from M2/M5-branes have correlation functions such as (1.3) as the operators $U(\Sigma_2)$ and $V(\Sigma_2)$ above simply pick up the flux sourced by these transverse to the non-compact wrapping loci. Whenever the M2/M5-branes constructing defects are wrapped on $\mathrm{Cone}(\gamma_1')$ then the flux through Γ_2 is determined by the intersection $\Gamma_2 \cdot \gamma_1'$.

[1]See [3] for further key discussion on flux operators.

Noting that the linking form on $H_1(S^3/\Gamma_{\text{ADE}};\mathbb{Z})$ is given by

$$\begin{aligned} \ell \, : \; H_1(S^3/\Gamma_{\text{ADE}};\mathbb{Z}) \otimes H_1(S^3/\Gamma_{\text{ADE}};\mathbb{Z}) \;&\to\; \mathbb{Q}/\mathbb{Z}\,, \\ (\gamma_1,\gamma_1') \;&\mapsto\; \ell(\gamma_1,\gamma_1') = \frac{1}{|\mathcal{Z}(\mathfrak{g})|}\,\Gamma_2\cdot\gamma_1'\,, \end{aligned} \tag{1.10}$$

we learn that, projected onto the defect group, (1.3) geometrizes as:

$$\langle \text{M5}(\Sigma_5,\gamma_1)\text{M2}(\Sigma_1,\text{Cone}(\gamma_1'))\dots\rangle = \exp\left(2\pi i \ell(\gamma_1,\gamma_1')\text{Link}(\Sigma_5,\Sigma_1)\right)\langle \text{M2}(\Sigma_1,\text{Cone}(\gamma_1'))\dots\rangle\,. \tag{1.11}$$

Here we indicated the symmetry operator supported on Σ_5 constructed by an asymptotic wrapping of an M5-brane over γ_1 by $\text{M5}(\Sigma_5,\gamma_1)$ and a Wilson line supported on Σ_1 constructed by the wrapping of an M2-brane on a non-compact cycle $\text{Cone}(\gamma_1')$ by $\text{M2}(\Sigma_1,\text{Cone}(\gamma_1'))$. We used that by $\mathbb{D}^{(1)}\cong\mathcal{Z}(\mathfrak{g})$ we know that the N-ality of the defect constructed from the M2-brane wrapping is $N_{\mathcal{R}}=1$. An analogous relation to the above hold upon interchanging M2's and M5's.

2 "Branes at Infinity"

The above discussion generalizes straightforwardly to a manifold of settings yielding constructions for topological operators.

Let X be a non-compact geometry such that $\mathbb{R}^{1,d-1}\times X$ constitutes the geometric profile of some consistent background of IIA/IIB string theory or M-theory. We consider "purely geometric" backgrounds, e.g., at this point we exclude the possibility of brane sources. Then we engineer the non-gravitational theory

$$X \xrightarrow{\text{IIA/IIB/M}} \mathcal{T}_X\,. \tag{2.1}$$

If the internal space is a metric cone $X=\text{Cone}(\partial X)$ over a smooth link ∂X then the engineered theory is d-dimensional after taking the standard geometric engineering limits and we write $\mathcal{T}_X^{(d)}$. More general situations are possible.

IIA/IIB/M-theory come with a diverse spectra of p-branes. Given an asymptotic cycle $\gamma_k\in \text{Tor}\,H_k(\partial X;\mathbb{Z})$ we produce, by wrapping a p-brane "at infinity" of the extra-dimensional geometry, the topological operator

$$\begin{aligned} \mathcal{N}(\Sigma_{p-k+1}) &= \int [D\Phi]\exp\left(2\pi i S^{\text{top}}[\Phi;\gamma_k\times\Sigma_{p-k+1}]\right) \\ &= \int [D\Phi]\exp\left(2\pi i S^{\text{top}}_{\gamma_k}[\Phi;\Sigma_{p-k+1}]\right)\,, \end{aligned} \tag{2.2}$$

where $S^{\text{top}}[\Phi;\gamma_k\times\Sigma_{p-k+1}]$ are contributions to the brane action from topological terms which result in $S^{\text{top}}_{\gamma_k}[\Phi;\Sigma_{p-k+1}]$ after performing all extra-dimensional integrals and Φ denotes all worldvolume fields of the brane. The topological operator $\mathcal{N}(\Sigma_{p-k+1})$ is a symmetry operator of $\mathcal{T}_X$ [4].

Let us make this explicit for the case of D-branes in IIA/IIB string theory. There, the relevant topological terms are the Wess-Zumino terms of the D-brane. With overall world volume M_{p+1} we explicitly have

$$S_{\mathrm{D}p}^{\mathrm{top}}[M_{p+1}] = \int_{M_{p+1}} \exp\left(f_2 - B_2^{\mathrm{NS}}\right) \sqrt{\frac{\widehat{A}(TM_{p+1})}{\widehat{A}(NM_{p+1})}} \bigoplus_{\mathrm{odd/even}} C_n^{\mathrm{RR}}\,, \tag{2.3}$$

where $f_2 = da_1$ is the field strength of the $U(1)$ gauge field on the brane worldvolume which here is the only degree of freedom path integrated over. The other fields are background fields pulled back onto the brane worldvolume they are the NS 2-form potential B_2^{NS} the supergravity RR-gauge fields

$$\bigoplus_{\mathrm{odd/even}} C_n^{\mathrm{RR}} = \begin{cases} C_1 + C_3 + C_5 + C_7 + C_9 & \text{IIA String Theory}\,, \\ C_0 + C_2 + C_4 + C_6 + C_8 & \text{IIB String Theory}\,, \end{cases} \tag{2.4}$$

and curvature terms associated with the tangent and normal bundle of the brane worldvolume known as the $\widehat{A}$-roof genus, which associates to some bundle E the sum of Pontryagin classes $p_i(E)$ according to

$$\widehat{A}(E) = 1 - \frac{1}{24}p_1(E) + \frac{1}{5760}\left(7p_1(E)^2 - 4p_2(E)\right) + \dots\,. \tag{2.5}$$

The expression (2.3) should be read as indicating that all functions are to be expanded in their respective power series, with cup-products replacing multiplication. From the fully expanded expression the integration is then against all terms of overall degree $p+1$. With this note, that we always have the electric top-degree coupling

$$S_{\mathrm{D}p}^{\mathrm{top}}[M_{p+1}] = \int_{M_{p+1}} C_{p+1} + \dots\,, \tag{2.6}$$

the other terms are necessary, in part, for consistencies with various anomalies and were historically indeed determined from anomaly inflow considerations.

Next, we make some important observations:

- Remarks on the extra-dimensional constructions of defect operators carry over. We highlight that symmetry operators are even simpler geometrically, with their construction depending exclusively on the asymptotic geometry via $\mathrm{Tor}\, H_k(\partial X;\mathbb{Z})$, rather than $\mathrm{Tor}\, H_k(\partial X;\mathbb{Z})|_{\mathrm{triv}}$. Further, they are of course also inherently topological in character. Their construction makes no reference to supersymmetric / holomorphic structures and is inherently non-Lagrangian (S^{top} is the action of the wrapped brane not the action of the engineered theory $\mathcal{T}_X$) applying directly to relative theories.

- Non-Invertible Symmetry Operators. In field theory, non-invertible symmetry operators support topological degrees of freedom along their world volume which lead to the non-invertible fusion relations. In string constructions, such localized degrees of freedom are automatically included and derive from the worldvolume physics of the wrapped brane.

- Algorithmic Construction. In higher-dimensional theories topological symmetry operators are often difficult to identify. In string constructions symmetry operators follow algorithmically from the "branes at infinity" construction.

- Generalizations. In the above we used p-branes in the "branes at infinity" construction, including D-branes of type II string theory and M-branes of M-theory. In this context, such membranes should be viewed as consistent supergravity singularities. As such above considerations are starting points in generalizing to other singular supergravity profiles, including metric singularities [5] and flux-branes [6].

- Applications to holography. Let us consider AdS/CFT. In extra-dimensional constructions, boundary CFT symmetry operators are derived from non-topological and non-perturbative extended objects (branes) of the gravitational bulk. Bulk reconstruction methods allow for an inversion of this approach: all symmetry operators of the CFT predict dual non-topological and non-perturbative bulk objects.

- Application to the no-global-symmetries conjecture in quantum gravity. Compactifying X, by viewing as a local patch in some larger compact geometry, couples $\mathcal{T}_X^{(d)}$ to d-dimensional gravity. We can now explicitly follow defect operators and symmetry operators through such geometric manipulations and observe compliancy with the no-global-symmetries conjecture.

We comment also that the fusion of symmetry operators is of great interest and the above setting related to the K-theory classifying brane charges [7].

3 Symmetry Operator Constructions via Example

We now exemplify the observations of the preceding section.

3.1 Example: Duality Defects

One core class of non-invertible symmetry operators in higher dimensions are duality defects. We review their field theoretic and top down construction for the example of 4D $\mathcal{N}=4$ $SU(2)$ supersymmetric Yang-Mills theory. See [8, 9] for the original discussions and further details.

We begin with the field theoretic discussion. The defect group of lines $\mathbb{D}^{(1)}$ of 4D $\mathcal{N}=4$ $\mathfrak{su}(2)$ SYM, consisting of Wilson and 't Hooft lines, is

$$\mathbb{D}^{(1)} \cong \mathbb{Z}_2^{(W)} \oplus \mathbb{Z}_2^{(H)} \,. \tag{3.1}$$

All three $\mathbb{Z}_2$ subgroups $\mathbb{Z}_2^{(W)}$, $\mathbb{Z}_2^{(H)}$, $\mathbb{Z}_2^{(W+H)}$ are maximal and mutually non-local and correspond to the global forms $SU(2), SO(3)_+, SO(3)_-$ respectively. All theories have a $\mathbb{Z}_2$ 1-form symmetry with background field B_2^0, B_2^+, B_2^- respectively. S-duality maps line operators as $S:(W,H)\mapsto(H,-W)$ and the partition functions transform according to

$$\begin{aligned} Z_{SO(3)_-}[\tau, B_2^-] &= \exp\left(\frac{i\pi}{2}\int \mathcal{P}(B_2^-)\right) Z_{SO(3)_-}[-1/\tau, S(B_2^-)]\,, \\ Z_{SU(2)}[\tau, B_2^0] &= Z_{SO(3)_+}[-1/\tau, S(B_2^0)]\,, \end{aligned} \tag{3.2}$$

which, when dropping reference to which polarization the background fields make reference to, i.e., simply viewing them as correctly oriented 2-form $\mathbb{Z}_2$ potentials with respect to $\mathbb{D}^{(1)}$, is equivalently

$$\begin{aligned} Z_{SO(3)_-}[\tau, B_2] &= \exp\left(\frac{i\pi}{2}\int \mathcal{P}(B_2)\right) Z_{SO(3)_-}[-1/\tau, B_2]\,, \\ Z_{SU(2)}[\tau, B_2] &= Z_{SO(3)_+}[-1/\tau, B_2]\,. \end{aligned} \tag{3.3}$$

Here $\mathcal{P}(B_2)$ denotes the Pontryagin square[2] of B_2. For the $SO(3)_-$ theory there is a mixed anomaly between S-duality and the 1-form symmetry. In the $SO(3)_-$ theory we can therefore consider the configuration

$$\mathcal{D}(M_3^+, B_2) \exp\left(\frac{i\pi}{2}\int_{M_3\times I} \mathcal{P}(B_2)\right) \mathcal{A}^{2,1}(M_3^-, B_2)\,, \tag{3.4}$$

in presence of a background field B_2. This configuration is defined with respect to the slab $M_3 \times I$ in spacetime with interval I and boundaries $\partial(M_3 \times I) = M_3^+ - M_3^-$. The operator $\mathcal{D}(M_3^+, B_2)$ realizes the anomalous S-duality transformation, the exponential in the bulk of the slab is a consequence of the anomaly. The highly non-trivial part is the 3D TFT $\mathcal{A}^{2,1}$ which exhibits the same anomalous transformations under gauge transformations of B_2 as $\mathcal{D}$ from which it is however distinct.

The expression (3.4) is invariant with respect to gauge transformations of the background B_2. One can therefore make it dynamical and change the global form to $SU(2)$. Contracting the slab and specializing to $\tau = i$, such that the theories in the half-spaces left and right of $M_3 = M_3^+ = M_3^-$ agree, one finds the non-invertible codimension-1 symmetry operator

$$\mathcal{N}(M_3) \equiv \mathcal{D}(M_3, b_2)\mathcal{A}^{2,1}(M_3, b_2)\,, \tag{3.5}$$

of the $SU(2)$ theory, where b_2 is now dynamical.

Let us now give an extra-dimensional construction of $\mathcal{N}$. We consider 4D $\mathcal{N} = 4$ $\mathfrak{su}(2)$ supersymmetric Yang-Mills theory as the worldvolume theory on a stack of 2 D3-branes in flat space and will neglect the center of mass $\mathfrak{u}_1$. The extra-dimensional geometry is the space transverse to the D3-branes, i.e., $X = \mathbb{R}^6$. The asymptotic boundary is $\partial X = S^5$.

We now wrap 7-brane of type III* (according to Kodaira's classification of elliptic surface singularities) on $M_3 \times S^5$ with M_3 along the D3-brane world volume. This 7-brane is a bound state of (p, q)-7-branes. It is a solution with constant axio-dilaton profile $\tau = i$ which also sets the corresponding gauge theory couplings, which we have also denoted τ. The key question now is: what is the 3D TFT living M_3 after reducing this 7-branes worldvolume physics on ∂X?

Without an explicit worldvolume Lagrangian, as in the case of D-branes, we cannot proceed simply via some reduction procedure and instead argue by studying the objects which can end on the 7-brane. Both F1- and D1-strings can end on the 7-brane resulting in line defect of the 3D TFT. Utilizing a dual M-theory frame we can compute their spins and match data to the analogous properties of the 1-form symmetry of the 3D TFT $\mathcal{A}^{2,1}(M_3)$. Here $\mathcal{A}^{N,p}(M_3)$ is the so-called minimal abelian TFT with $\mathbb{Z}_N$ 1-form symmetry and label p [10].

[2] The Pontryagin square is an operation in cohomology $\mathcal{P} : H^2(X, \mathbb{Z}_N) \to H^4(X, \mathbb{Z}_{\gcd(2,N)N})$ such that $\mathcal{P}(B) = B \cup B$ mod N, if lifts exist, e.g., there is a commutative diagram with $\widetilde{\mathcal{P}} : H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z})$ then $\widetilde{\mathcal{P}}$ is the cup product, if N is even and X is spin then $\mathcal{P}$ maps onto the $\mathbb{Z}_N$ subgroup of $\mathbb{Z}_{2N}$.

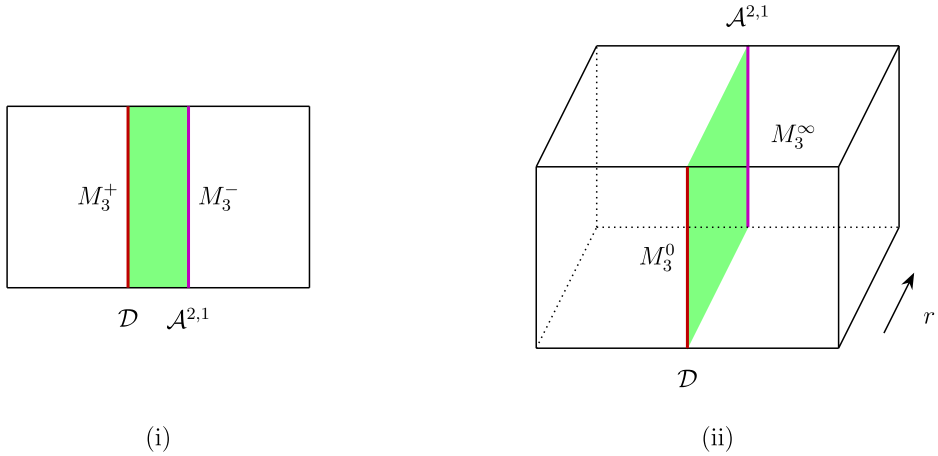


Figure 3: (i): Sketch of the decompression of the non-invertible symmetry operator $\mathcal{N}$. Decompression is with respect to a spacetime coordinate as in (3.4). (ii) Sketch of the extra-dimensional geometry after compactification on ∂X. The 7-brane wrapping results in the TFT $\mathcal{A}^{2,1}$. The decompression is naturally realized along the radial coordinate in X. In 4D compressing the configuration in (i) agrees with the configuration constructed in (ii).

Next, we discuss the background 2-form fields. In IIB we have B_2^{NS} and C_2^{RR} forming an $\mathrm{SL}(2;\mathbb{Z})$ doublet. They are mapped as $(B_2^{\mathrm{NS}}, C_2^{\mathrm{RR}}) \mapsto (C_2^{\mathrm{RR}}, -B_2^{\mathrm{NS}})$ under S-duality. Consequently, once the 7-brane is inserted, we can no longer speak of two distinct 2-form profiles, the $\mathrm{SL}(2;\mathbb{Z})$ monodromy combines them into a joint quantity - they still make sense locally, but are no longer globally defined 2-form gauge potentials. At the III^* 7-brane, which sources the monodromy, only eigenvectors to the monodromy action can be non-vanishing and in principle couple to the degrees of freedom localized there. After compactifying on S^5 the fields B_2^{NS} and C_2^{RR} take values in $\mathbb{Z}_2$ (due to the D3-brane 5-form flux) and the TFT on M_3 can therefore only couple to the linear combination

$$B_2^- = B_2^{\mathrm{NS}} + C_2^{\mathrm{RR}} . \tag{3.6}$$

The 2-form $B_2^{\mathrm{NS}}, C_2^{\mathrm{RR}}$ couples to F1- and D1-strings, which stretched between the D3-brane stack and ∂X become Wilson lines and 't Hooft lines respectively. Matching to field theory we therefore have $B_2^{\mathrm{NS}} = B_2^0$ while $C_2^{\mathrm{RR}} = B_2^+$. Their sum is then identified with B_2^- as above.

Next, we turn to discuss the other terms of (3.4). The key insight is that there is a completely analogous separation of terms over a slab also in the extra-dimensional setup. The only difference being that I is now the internal radial direction of X. Consequently, to a 4D observer $\mathcal{D}$ and $\mathcal{A}^{2,1}$ will appear stacked, immediately resulting in $\mathcal{N}$. See figure 3.

The operator $\mathcal{D}(M_3)$ lives on a codimension-1 surface M_3 in the world volume of the D3-brane stack. Pushing it across local operators they transform according to an S-duality transformation and this is its defining property. In the extra-dimensional geometry consistency of this configuration can be phrased as saying that $\mathcal{D}(M_3)$ must live at the boundary of an $\mathrm{SL}(2;\mathbb{Z})$ branch cut.

In summary, the III^* 7-brane wrapped on ∂X constructs $\mathcal{A}^{2,1}(M_3^{\infty}, B_2^-) \otimes \mathcal{T}_{\mathrm{3D}}$ where $\mathcal{T}_{\mathrm{3D}}$ may be some non-trivial theory, which however is fully decoupled from the 4D physics, and which can therefore be negelected. The cutting and gluing / transition function prescription on the world D3-brane stack worldvolume defined $\mathcal{D}(M_3^0, B_2^-)$. Here the exponents $0, \infty$ make reference to extra-dimensional location of the 3-surfaces M_3 with respect to the radial coordinate r of X. The D3-branes are at $r = 0$ while the asymptotic boundary is at $r = \infty$. After compactification on S^5, we can localize the $\mathrm{SL}(2;\mathbb{Z})$ configuration sourced by these objects onto a branch cut. This branch cut can be shown to support $\exp\left(\frac{i\pi}{2}\int \mathcal{P}(B_2^-)\right)$. More generally, there is a term proportial to $A_1 \cup \mathcal{P}(B_2^-)$ in the 5D bulk and the branch cut simply corresponds to gauging A_1 to a delta-function profile with respect to the two sources $\mathcal{D}, \mathcal{A}^{2,1}$.

See [11] for further details and generalizations to large classes of 4D $\mathcal{N} = 1$ SCFTs.

3.2 Example: Bulk Reconstruction in Holography

We now discuss some key considerations motivated by the "branes at infinity" construction.

Let us now consider symmetry operators in the context of the AdS/CFT correspondence. There, the holographic dictionary asserts that every boundary CFT operators has a dual gravitational bulk AdS counterpart. What are the bulk counterparts to topological CFT symmetry operators? Consistency with the no-global-symmetries conjecture in quantum gravity requires these bulk objects to be non-topological, as any topological operator is interpreted as generating a symmetry [2].

This question should be first approached in the setting of string holography. There the gravitational bulk theory contains p-branes and in the bulk these branes are non-topological as characterized by their tension (i.e., coupling to gravity in a probe approximation) and as specified by their non-topological world volume degrees of freedom. We can now consider pushing such branes to the conformal boundary. On the brane worldvolume this realizes a limit localizing us to topological field configurations. The brane reduces to a symmetry operator which is computed from the extra-dimensional wrapping locus via compactification following analogous steps as in (2.2). For further discussion and first examples see [12, 13].

This sets the general expectation: Any topological CFT symmetry operator, when pushed from the conformal boundary into the bulk, ought to come to life as a non-topological bulk "brane".

The existence of this "brane" and that it must be non-topological can be argued via a proof of contradiction employing bulk reconstruction methods, starting from first observations in [14].

In order to expand on this, we will take our starting point in the Euclidean setting, i.e., we have a Euclidean CFT and the radial direction r of the Euclidean AdS can be viewed as RG time. On the CFT boundary we consider the field theory symmetry operator $\mathcal{N}$ linking a charged operator $\mathcal{O}$. If we were to contract $\mathcal{N}$ onto $\mathcal{O}$ we would obtain the symmetry transformed operator $\mathcal{O}^{(\mathcal{N})}$. We now solve the bulk equations of motion with respect to these sources radially inwards. This is achieved in part via convolution with a smearing kernel which can be interpreted as RG flow. The deeper we proceed into the AdS bulk the more the bulk duals $\widetilde{\mathcal{O}}$ and $\widetilde{\mathcal{N}}$ (if it exists) smear together as their separation can be resolved less and less. Ultimately, deep in the bulk, they resemble the bulk reconstruct $\widetilde{\mathcal{O}}^{(\mathcal{N})}$. Overall, this implies the existence of a crossover region at some characteristic depth r_*. See subfigure (i) of figure 4.

Assume now that $\mathcal{N}$ does not have a bulk counter part, i.e., $\widetilde{\mathcal{N}}$ does not exist. Then the above describes a procedure where the bulk reconstruct of $\mathcal{O}$ transitions to the bulk reconstruct of $\mathcal{O}^{(\mathcal{N})}$. Compared with the bulk reconstruct of $\mathcal{O}$ with no insertion of $\mathcal{N}$ on the boundary, see subfigure (ii)

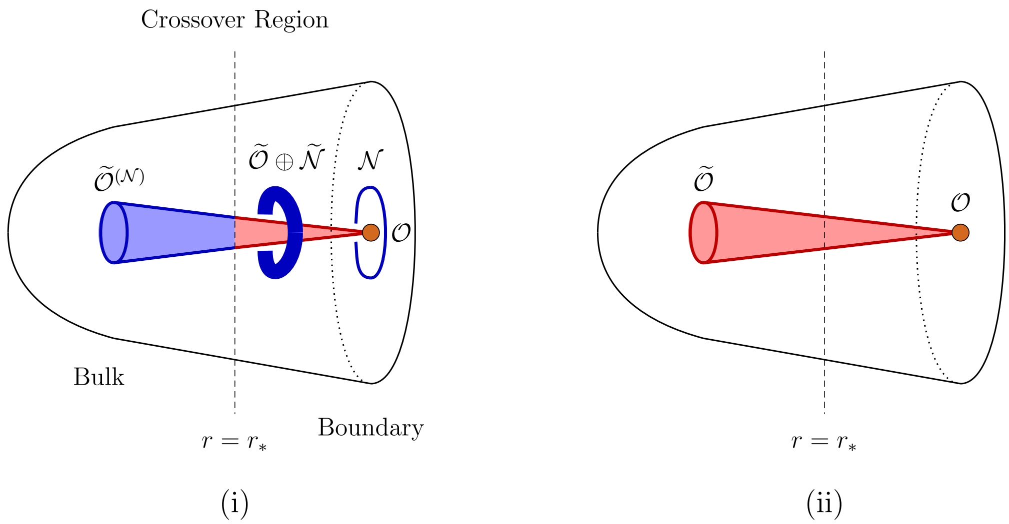


Figure 4: We sketch bulk reconstructed field configurations starting from boundary configurations. In (i) we sketch the bulk reconstruct with insertions of the CFT symmetry operator $\mathcal{N}$ and charged operator $\mathcal{O}$. When $r \gg r_*$ they separately smear to $\widetilde{\mathcal{O}}$ and $\widetilde{\mathcal{N}}$. By $r \ll r_*$ they have smeared to $\widetilde{\mathcal{O}}^{(\mathcal{N})}$. In (ii) we sketch the bulk reconstruct with only $\mathcal{O}$ alone.

of figure 4, this configuration exhibits distinct bulk stress energy localized to the crossover region, simply due to non-vanishing derivatives in the radial profile. An important detail here is that this difference can be argued to be physical, in no sense are $\widetilde{\mathcal{O}}^{(\mathcal{N})}$ and $\widetilde{\mathcal{O}}$ gauge equivalent in the bulk.

The jump in profile before and after r_* allows us determine that there must be something else present besides just $\widetilde{\mathcal{O}}$ for $r \gg r_*$ and $\widetilde{\mathcal{O}}^{(\mathcal{N})}$ for $r \ll r_*$ (where the conformal boundary is at $r = \infty$). This establishes the existence of $\widetilde{\mathcal{N}}$. Further, the stress energy localized to the crossover region is due to $\widetilde{\mathcal{N}}$ which can therefore not be topological. For further discussion see [15].

The above argument is non-constructive and explicit examples of $\widetilde{\mathcal{N}}$ have so far been constructed on an example by example basis. Here, we mention the example of "metric branes" where the bulk object of interest, dual to a topological CFT symmetry operator, is a particular metric curvature singularity defect in supergravity.

To discuss this example in more depth we consider IIB supergravity on $\mathrm{AdS}_5 \times S^5$ with N units of 5-form flux on S^5. This semi-classical background is AdS/CFT dual to a stack of N D3-branes. The dual field theory exhibits an $SU(4)$ R-symmetry which is the double covering of the $SO(6)$ isometry group of the normal geometry to the D3-brane stack. We ask, what are the gravitational dual objects to topological R-symmetry operators?

We will be brief. First, as the symmetry operator is supported on some 3-surface in the CFT spacetime, the gravitational bulk object will be some 3-brane. This brane is a codimension-2 object characterized in part by its monodromy action on S^5. The bulk object dual to a symmetry operator labelled by some $g \in SO(6)$ is such that along path linking the brane we have $S^5 \mapsto g \cdot S^5$. The full $SU(4)$ is seen upon extending this action to all supergravity fields, not just the metric on which we will focus here.

This monodromy characterization is completely parallel to our discussion of duality defects in the preceding section - which was characterized by an $\mathrm{SL}(2;\mathbb{Z})$ monodromy.[3] First, the action of the R-symmetry on local operators can be understood as the brane connecting to the conformal boundary along a branch cut terminating on some analog to the operator $\mathcal{D}$. Second, the monodromy forces a degenerate fiber in the S^5 fibration over AdS_5. This degeneration leads in many cases to spindle-like solutions [16, 17] which fully characterizes the brane as a particular singular metric defect.

3.3 Example: No-Global-Symmetries Conjecture

Viewing the non-compact geometry X as some local patch $X = X^{\mathrm{loc}}$ of a compact geometry

$$Y = X^{\mathrm{loc}} \cup X^{\circ}\,, \tag{3.7}$$

we can realize a completion of the initial non-gravitational theory $\mathcal{T}_X$ to a theory of quantum gravity as specified via Y. By the no-global-symmetries conjecture, see e.g., the discussion in [18, 19], we expect that defect operators do not remain non-dynamical and that symmetry operators do not remain topological under this compactification. Ultimately all symmetries of $\mathcal{T}_X$ must be broken or gauged. Conversely, we can ask how does the quantum gravity theory associated to Y comply with the no-global-symmetries conjecture. What happens to all the symmetries that arise by taking local limits isolating QFT degrees of freedom localized in Y?

Of course there are no non-compact cycles in Y to construct defect operators and no asymptotic cycles to construct topological symmetry operators via p-brane wrappings, and therefore the main question becomes to geometrically characterize the gauging and breaking of symmetries of $\mathcal{T}_X$.

Let us do so in an explicit example. Consider M-theory on $\mathbb{R}^{1,6} \times T^4/\mathbb{Z}_2$. Here the $\mathbb{Z}_2$ acts via reflection on each of the four torus circles such that north and south pole are fixed. The torus T^4 therefore has $2^4 = 16$ fixed points and the quotient $T^4/\mathbb{Z}_2$ has 16 singularities which are locally modeled on $\mathbb{C}^2/\mathbb{Z}_2$. The orbifold $T^4/\mathbb{Z}_2$ is a K3-surface and consequently we are considering a 7D supergravity theory.

To proceed we define the disjoint union of local patches

$$X^{\mathrm{loc}} = \sqcup_{i=1}^{16} X_i^{\mathrm{loc}}\,, \tag{3.8}$$

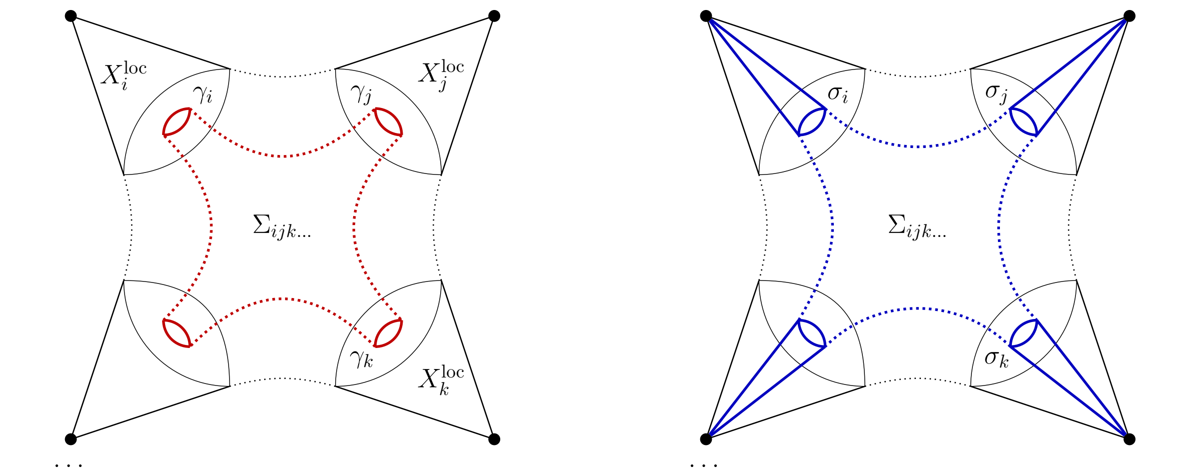
where each X_i^{loc} is modeled on $\mathbb{C}^2/\mathbb{Z}_2$. By the discussion in the previous lecture X^{loc} supports the localized degrees of freedom of a supersymmetric 7D Yang-Mills theory with gauge group

$$\mathfrak{g} = \mathfrak{su}(2)^{\oplus 16}\,. \tag{3.9}$$

We therefore have an overall defect group of Wilson lines and 't Hooft 4-surface operators as

$$\mathbb{D} = \left(\mathbb{Z}_2^{(W)} \oplus \mathbb{Z}_2^{(H)}\right)^{\oplus 16}\,, \tag{3.10}$$

[3]This parallel can be made completely explicit when constructing 4D $\mathcal{N} = 4$ SYM in IIB string theory via the purely geometric background $\mathbb{R}^{1,3} \times T^2 \times \mathbb{C}^2/\Gamma_{\mathrm{ADE}}$. In this case the $\mathrm{SL}(2;\mathbb{Z})$ monodromy acts on the torus T^2 and the duality defect becomes a geometric singularity given by a particular torus degeneration, in contrast to the construction via a III* 7-brane when the theory is realized on a stack of D3-branes.



(i) : Trivialization of Symmetry Operators $\qquad$ (ii) : Compactification of Defect Operators

Figure 5: Both subfigures (i) and (ii) show a singular K3 surface Y and as subsets thereof the balls $X_i^{\rm loc} \subset Y$ (black cones). Subfigure (i) shows cycles $\gamma_i \in H_1(\partial X_i^{\rm loc};\mathbb{Z})$ supporting symmetry operators (blue) which are bounded by a bulk 2-chain $\Sigma_{ijk\dots}$. Subfigure (ii) shows cycles $\sigma_i \in H_2(X_i^{\rm loc},\partial X_i^{\rm loc};\mathbb{Z})$ with boundaries (blue) compactified by the same 2-chain $\Sigma_{ijk\dots}$ to a compact curve in Y.

constructed from M2- and M5-branes respectively and acted on by Gukov-Witten operators and magnetic flux operators also constructed from M5- and M2-branes respectively. In the geometry $X^{\rm loc}$ both defect and symmetry operators are associated with (cones over) classes in

$$H_1(\partial X^{\rm loc}) = \bigoplus_{i=1}^{16} H_1(S^3/\mathbb{Z}_2) \cong \mathbb{Z}_2^{16}\,. \tag{3.11}$$

The space X° is then constructed from $Y = T^4/\mathbb{Z}_2$ by excising 16 non-overlapping balls centered on the singularities.

The gauging and breaking of symmetries, and the coupling of $\partial X^{\rm loc}$ to X° is then informed solely by the exact sequence

$$0 \;\to\; H_2(X^\circ;\mathbb{Z}) \;\to\; H_2(Y;\mathbb{Z}) \;\to\; H_1(\partial X^{\rm loc};\mathbb{Z}) \;\to\; H_1(\partial X^\circ;\mathbb{Z}) \;\to\; 0\,, \tag{3.12}$$

and compliancy with the no-global-symmetries conjecture will be equivalent to its exactness. This sequence arises as the simplification of an exact subsequence contained in the long exact Mayer-Vietoris sequence to the covering $Y = \partial X^{\rm loc} \cup X^\circ$. Entry by entry the sequence evaluates as

$$0 \;\to\; \mathbb{Z}^6 \;\to\; \mathbb{Z}^6 \oplus \mathbb{Z}_2^5 \;\to\; \mathbb{Z}_2^{16} \;\to\; \mathbb{Z}_2^5 \;\to\; 0\,, \tag{3.13}$$

where the mapping $\mathbb{Z}^6 \to \mathbb{Z}^6$ is multiplication by 2.

Let us discuss the physics of this sequence. The group $H_2(X^\circ;\mathbb{Z})$ informs us that coupling in X° adds extra abelian gauge degrees of freedom to our effective physics such that the overall gauge algebra is updated to

$$\mathfrak{g}' = \mathfrak{su}(2)^{\oplus 16} \oplus \mathfrak{u}(1)^{\oplus 6}\,. \tag{3.14}$$

The image, kernel of the mapping $H_1(\partial X^{\text{loc}};\mathbb{Z}) \to H_1(\partial X^\circ;\mathbb{Z})$ characterizes the 1-cycles which remain independent, trivialize in the bulk X° respectively. The trivializing 1-cycles inform us about the compact 2-cycles added to the geometry. In the sequence this manifests as exactness, i.e., the kernel of $H_1(\partial X^{\text{loc}};\mathbb{Z}) \to H_1(\partial X^\circ;\mathbb{Z})$ and the image of $H_2(Y;\mathbb{Z}) \to H_1(\partial X^{\text{loc}};\mathbb{Z})$ agree. These compact 2-cycles can be wrapped by M2-branes contributing matter. The mapping $H_2(X^\circ;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$ contains some interesting normalization data on the charges carried by these states under the extra $\mathfrak{u}_1^{\oplus 6}$ factor. One consequence, which we will not explain here, is that there is an identification of a $\mathbb{Z}_2^6$ subgroup of the corresponding $U(1)^6$ with a particular $\mathbb{Z}_2^6$ center subgroups of $SU(2)^{16}$. See figure 5 for a sketch of the key geometric effects discussed above.

Finally, let us now discuss the gauging, breaking and the latter identification in the context of the no-global-symmetries conjecture. To begin, pick an electric polarization and assume we start with an $SU(2)^{16}$ gauge theory in the local model. Then we have a $\mathbb{Z}_2^{16}$ 1-form center symmetry group. Now couple in X°. Via the identifications the gauge group becomes

$$\left(SU(2)^{16} \times U(1)^6\right)/\mathbb{Z}_2^6\,, \tag{3.15}$$

and of the original $\mathbb{Z}_2^{16}$ center symmetry only a $\mathbb{Z}_2^{10}$ subgroup remains. We have also added in massive matter via M2-brane wrapping of compact curves. This breaks the center symmetry further to $\mathbb{Z}_2^5$. This subgroup is not broken, and must therefore be gauged, the gauge group becomes

$$G = \left(\left(SU(2)^{16}/\mathbb{Z}_2^5\right) \times U(1)^6\right)/\mathbb{Z}_2^6\,, \tag{3.16}$$

and a dual $\mathbb{Z}_2^5$ 4-form symmetry arises. This 4-form symmetry is broken by states engineered via M5-brane wrappings on compact curves. Overall none of the original symmetries survive.

See [5] for further examples and discussion.

References

[1] A. Hatcher, *Vector Bundles and K-Theory.* 2003. http://www.math.cornell.edu/˜hatcher.

[2] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, "Generalized Global Symmetries," *JHEP* **02** (2015) 172, arXiv:1412.5148 [hep-th].

[3] I. n. García Etxebarria, B. Heidenreich, and D. Regalado, "IIB flux non-commutativity and the global structure of field theories," *JHEP* **10** (2019) 169, arXiv:1908.08027 [hep-th].

[4] J. J. Heckman, M. Hübner, E. Torres, and H. Y. Zhang, "The Branes Behind Generalized Symmetry Operators," *Fortsch. Phys.* **71** no. 1, (2023) 2200180, arXiv:2209.03343 [hep-th].

[5] M. Cvetič, J. J. Heckman, M. Hübner, and E. Torres, "Generalized symmetries, gravity, and the swampland," *Phys. Rev. D* **109** no. 2, (2024) 026012, arXiv:2307.13027 [hep-th].

[6] M. Cvetič, J. J. Heckman, M. Hübner, and E. Torres, "Fluxbranes, generalized symmetries, and Verlinde's metastable monopole," *Phys. Rev. D* **109** no. 4, (2024) 046007, arXiv:2305.09665 [hep-th].

[7] I. Bah, E. Leung, and T. Waddleton, "Non-invertible symmetries, brane dynamics, and tachyon condensation," *JHEP* **01** (2024) 117, arXiv:2306.15783 [hep-th].

[8] J. Kaidi, K. Ohmori, and Y. Zheng, "Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories," *Phys. Rev. Lett.* **128** no. 11, (2022) 111601, arXiv:2111.01141 [hep-th].

[9] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, "Noninvertible duality defects in 3+1 dimensions," *Phys. Rev. D* **105** no. 12, (2022) 125016, arXiv:2111.01139 [hep-th].

[10] P.-S. Hsin, H. T. Lam, and N. Seiberg, "Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d," *SciPost Phys.* **6** no. 3, (2019) 039, arXiv:1812.04716 [hep-th].

[11] J. J. Heckman, M. Hubner, E. Torres, X. Yu, and H. Y. Zhang, "Top down approach to topological duality defects," *Phys. Rev. D* **108** no. 4, (2023) 046015, arXiv:2212.09743 [hep-th].

[12] I. n. García Etxebarria, "Branes and Non-Invertible Symmetries," *Fortsch. Phys.* **70** no. 11, (2022) 2200154, arXiv:2208.07508 [hep-th].

[13] F. Apruzzi, I. Bah, F. Bonetti, and S. Schafer-Nameki, "Noninvertible Symmetries from Holography and Branes," *Phys. Rev. Lett.* **130** no. 12, (2023) 121601, arXiv:2208.07373 [hep-th].

[14] D. Harlow and H. Ooguri, "Symmetries in quantum field theory and quantum gravity," *Commun. Math. Phys.* **383** no. 3, (2021) 1669–1804, arXiv:1810.05338 [hep-th].

[15] J. J. Heckman, M. Hübner, and C. Murdia, "On the holographic dual of a topological symmetry operator," *Phys. Rev. D* **110** no. 4, (2024) 046007, arXiv:2401.09538 [hep-th].

[16] P. Bomans and L. Tranchedone, "Holographic generalised Gukov-Witten defects," *JHEP* **03** (2025) 118, arXiv:2410.18172 [hep-th].

[17] I. Arav, J. P. Gauntlett, Y. Jiao, M. M. Roberts, and C. Rosen, "Superconformal monodromy defects in $\mathcal{N}$=4 SYM and LS theory," *JHEP* **08** (2024) 177, arXiv:2405.06014 [hep-th].

[18] M. van Beest, J. Calderón-Infante, D. Mirfendereski, and I. Valenzuela, "Lectures on the Swampland Program in String Compactifications," *Phys. Rept.* **989** (2022) 1–50, arXiv:2102.01111 [hep-th].

[19] N. B. Agmon, A. Bedroya, M. J. Kang, and C. Vafa, "Lectures on the string landscape and the Swampland," arXiv:2212.06187 [hep-th].