

Lecture 3: Symmetry Theories in Geometric Engineering and Holography

Lecturer: Max Hübner (Harvard University and Uppsala University), max.elliott.huebner@gmail.com

Location: Shanghai Institute for Mathematics and Interdisciplinary Sciences (SIMIS)

Date: Tuesday and Thursday, June 17th and 19th 2025

Abstract: We discuss extra-dimensional constructions of symmetry theories and their utility.

1 Introduction and Motivating Examples

In the last two lectures we introduced defect operators and, acting on these, symmetry operators. We focussed on non-gravitational systems with extra-dimensional constructions in string theory via some geometry X and constructed both sets of operators from p -branes and cycles of the asymptotic boundary ∂X .

In recent years it has been appreciated that many features of symmetries in quantum field theory can be isolated into symmetry theories. The simplest of which is the so-called symmetry TFT, an auxiliary topological systems in one extra dimension compared to spacetime. In this lecture we discuss how this extra-dimensional construction derives from the extra-dimensional construction based on X .

1.1 Symmetries of 7D Yang-Mills Theory

We begin with a field theoretic discussion of symmetry theories. Consider 7D Yang-Mills theory with simply-laced gauge algebra \mathfrak{g} of type ADE in Cartan's classification and action

$$S_{7D}^{\text{YM}}[\mathfrak{g}] = -\frac{1}{4g_{\text{YM}}^2} \int_{7D} \text{Tr}(F_2 \wedge *F_2), \quad (1.1)$$

with field strength $F = dA + A \wedge A$ and connection 1-form A valued in the adjoint representation of the Lie algebra \mathfrak{g} . This theory has Wilson line and 't Hooft 4-surface defects

$$\mathbb{D}^{(1)} \cong \mathcal{Z}(\mathfrak{g}), \quad \mathbb{D}^{(4)} \cong \mathcal{Z}(\mathfrak{g}), \quad (1.2)$$

which are acted on by 1-form and 4-form symmetries generated by topological Gukov-Witten operators and magnetic flux operators. We denote the discrete background fields of these symmetries by B_2 and B_5 respectively. We model these discrete backgrounds via continuous fields with quantized periods, i.e., the periods are constrained to take values in $\mathcal{Z}(\mathfrak{g})$. Next, note that this gauge theory also has a continuous $U(1)$ 3-form instanton symmetry with symmetry operator

$$I_\alpha(M_4) = \exp\left(\frac{2\pi i\alpha}{8\pi^2} \int_{M_4} \text{Tr}(F_2 \wedge F_2)\right), \quad (1.3)$$

where $\alpha \in [0, 1)$ which acts on instantonic 3-surfaces which couple electrically to the abelian background field C_3 with field strength $G_4 = dC_3$. For non-trivial backgrounds C_3 we must therefore

extend our 7D action to include the term

$$S_{7D}^{\text{inst}}[\mathfrak{g}] = \frac{2\pi}{16\pi^3} \int_{7D} C_3 \text{Tr}(F_2 \wedge F_2), \quad (1.4)$$

such that, at this point, our overall action is $S_{7D}^{\text{YM}}[\mathfrak{g}] + S_{7D}^{\text{inst}}[\mathfrak{g}]$. However, as will momentarily become important, as C_3 is not globally well-defined (1.4) should be replaced with

$$S_{8D}^{\text{inst}}[\mathfrak{g}] = \frac{2\pi}{16\pi^3} \int_{8D} G_4 \text{Tr}(F_2 \wedge F_2), \quad (1.5)$$

integrated over a manifold with boundary, where the boundary is 7D spacetime.

Interestingly, the instanton 3-form symmetry and center 1-form symmetry are not independent. To see this, first consider the intersection of two topological Gukov-Witten operators. Individually these are supported in codimension-2 and their intersection is therefore of codimension-4, i.e., in 7D they intersect generically along 3-surfaces. Next, consider an insertion of $I_\alpha(M_4)$ transverse to this 3-surface, filling all of the $4 = 2 + 2$ normal dimensions. The two Gukov-Witten operators individually fill 2 of these 4 dimensions. Each carries magnetic flux picked up by one factor in $\text{Tr}(F_2 \wedge F_2)$ entering the operator $I_\alpha(M_4)$. Said differently, we find the 3-surface intersection of two Gukov-Witten operators to carry instanton charge Q_{inst} .

The 3-surface intersection is therefore expected to couple electrically to C_3 and therefore, when backgrounds are turned on, we are required to extend our 8D action by the coupling

$$S_{8D}^{\text{top}}[\mathfrak{g}] = 2\pi i Q_{\text{inst}} \int_{8D} \frac{G_4}{2\pi} \left(\frac{B_2}{2\pi} \right)^2. \quad (1.6)$$

in our 8D action. The corresponding 7D term proportional to $C_3 B_2^2$ corresponds manifestly to the intersecting Gukov-Witten operators in a gauge for the background B_2 where these are presented as the δ -function 2-forms Poincaré dual to the supports of the two Gukov-Witten operators.

The coupling (1.6) derives from the non-topological coupling (1.5). See [1, 2] for further details on the computation to follow. To make this more precise let us specialize to $\mathfrak{g} = \mathfrak{su}(N)$. Then, we can embed¹ the $\mathfrak{su}(N)$ gauge field A_1 into a $\mathfrak{u}(N)$ gauge field \tilde{A}_1 by writing

$$\tilde{A}_1 = A_1 + \frac{1}{N} \text{Id}^N A'_1, \quad (1.7)$$

where A' is a background \mathfrak{u}_1 gauge field that satisfies $B_2 = dA'_1/N$ and Id^N is the identity matrix. The 1-form center symmetry is then $(A_1, B_2) \rightarrow (A_1 + \lambda_1, B_2 + d\lambda_1)$. When a general background B_2 is turned on the coupling (1.5) depends on the auxiliary 8D bulk as

$$S_{8D}^{\text{inst}}[\mathfrak{su}(N)] \supset \frac{2\pi}{16\pi^3} \int_{8D} G_4 \text{Tr} \left[(\tilde{F}_2 - B_2 \text{Id}^N) \wedge (\tilde{F}_2 - B_2 \text{Id}^N) \right], \quad (1.8)$$

and is now dependent on the 8D auxiliary manifold extending spacetime. Each term involves both background fields and gauge fields or only background fields. Respectively, these indicate gauge

¹This realizes the coupling of the original system to a TFT whose fields include the background B_2 by making it manifest as a $\mathfrak{u}(1)$ field strength.

anomalies and global anomalies / 't Hooft anomalies. The latter will be quantified by Q_{inst} . To separate the global anomalies from the gauge anomalies note that the latter are quantified via Chern classes. The second Chern class evaluates as

$$c_2 = \frac{1}{8\pi^2} \left[\left(\text{Tr } \tilde{F} \right)^2 - \text{Tr} \left(\tilde{F} \wedge \tilde{F} \right) \right] = \frac{1}{8\pi^2} \left[N^2 B_2 \wedge B_2 - \text{Tr} \left(\tilde{F} \wedge \tilde{F} \right) \right], \quad (1.9)$$

and using this we have

$$\begin{aligned} \text{Tr} \left[\left(\tilde{F}_2 - B_2 \text{Id}^N \right) \wedge \left(\tilde{F}_2 - B_2 \text{Id}^N \right) \right] &= \text{Tr} \left(\tilde{F}_2 \wedge \tilde{F}_2 \right) - N B_2 \wedge B_2 \\ &= -8\pi^2 c_2 - N(N-1) B_2 \wedge B_2. \end{aligned} \quad (1.10)$$

Substituting this result into (1.8) then leads to the gauge and global anomalies

$$\begin{aligned} S_{\text{8D}}^{\text{gauge}}[\mathfrak{su}(N)] &= -2\pi i \int_{\text{8D}} \frac{G_4}{2\pi} c_2, \\ S_{\text{8D}}^{\text{global}}[\mathfrak{su}(N)] &= -2\pi i \frac{N(N-1)}{2} \int_{\text{8D}} \frac{G_4}{2\pi} \left(\frac{B_2}{2\pi} \right)^2, \end{aligned} \quad (1.11)$$

with $Q_{\text{inst}} = -N(N-1)/2$. The gauge anomaly constrains $G_4/2\pi$ and c_2 periods to be integral.

1.2 Symmetry Topological Field Theory of 7D Yang-Mills Theory

We turn to formulate the above discussion using symmetry topological field theories (SymTFTs). The goal of the SymTFT framework is to isolate the symmetry data from any additional data. One key insight underlying the framework is that this symmetry data carries structure, it organizes into a topological field theory and boundary conditions thereto.

In the SymTFT framework an absolute QFT is presented as a relative QFT $|Z\rangle$ contracted with a boundary condition $\langle B|$ to give the partition function of the system

$$\langle B|Z\rangle = Z(B), \quad (1.12)$$

These bra-kets are (co)vectors in a Hilbertspace of a topological field theory, the so-called SymTFT, living in one dimension higher than the QFT spacetime M . This wavefunction interpretation is made precise by decompressing the QFT along an auxiliary interval I . The SymTFT then lives on the slab $M \times I$. This slab has two boundaries, one of which supports the physical boundary condition $|Z\rangle$ while the other supports the topological boundary condition $\langle B|$. These are boundary conditions to the fields of the SymTFT filling the bulk of the slab. The decompressed setup is equivalent to the original absolute QFT and manifestly mapped onto it by contracting I .

Let us consider the example of 7D Yang-Mills Theory from section 1.1. We will simply state the result and then discuss its features. The SymTFT lives in 8D, its action is

$$S_{\text{8D}}^{\text{TFT}}[\mathfrak{g}] = 2\pi i \int_{\text{8D}} \left[|\mathcal{Z}(\mathfrak{g})| \frac{B_2}{2\pi} \frac{dB_5}{2\pi} + \frac{H_4}{2\pi} \frac{dC_3}{2\pi} + Q_{\text{inst}} \frac{dC_3}{2\pi} \left(\frac{B_2}{2\pi} \right)^2 \right]. \quad (1.13)$$

Here B_2, B_5, C_3 are gauge potentials valued in $U(1)$ while the gauge potential H_4 is valued in \mathbb{R} . Locally these are all 1-forms fields valued in \mathbb{R} . Globally they are distinguished via their sets of gauge transformations.² The equations of motion are

$$\begin{aligned}
0 &= |\mathcal{Z}(\mathfrak{g})| \frac{dB_2}{2\pi}, \\
0 &= \frac{dC_3}{2\pi}, \\
0 &= \frac{dH_4}{2\pi} + Q_{\text{inst}} d \left(\frac{B_2}{2\pi} \right)^2, \\
0 &= |\mathcal{Z}(\mathfrak{g})| \frac{dB_5}{2\pi} + 2Q_{\text{inst}} \frac{dC_3}{2\pi} \frac{B_2}{2\pi}.
\end{aligned} \tag{1.14}$$

Integrating these over manifolds with and without boundaries give rise respectively to the operators

$$\begin{aligned}
U_n &= \exp \left(in \int_{\Sigma_2} B_2 \right), \\
T_k &= \exp \left(ik \int_{\Sigma_3} C_3 \right), \\
W_\alpha &= \exp \left(i\alpha \int_{\Sigma_4} \left(H_4 + \frac{Q_{\text{inst}}}{2\pi} B_2^2 \right) \right), \\
V_n &= \exp \left(in \int_{\partial\Sigma_6} B_5 + \frac{2inQ_{\text{inst}}}{2\pi|\mathcal{Z}(\mathfrak{g})|} \int_{\Sigma_6} B_2 dC_3 \right),
\end{aligned} \tag{1.15}$$

where $n = 0, \dots, |\mathcal{Z}(\mathfrak{g})| - 1$ and $k \in \mathbb{Z}$ and $\alpha \in U(1)$. The manifolds $\Sigma_2, \Sigma_3, \Sigma_4$ are closed while the manifold Σ_6 has boundary $\partial\Sigma_6$. Relations between operators are worked out via canonical quantization starting from (1.13).

The action (1.13) is such that B_2, B_5 and C_3, H_4 are conjugate variables. This non-commutativity reflects the mutual non-locality as relevant in the choices of polarization. Concretely, this means that boundary conditions $\langle B|$ are constrained such that if we chose, for example, Dirichlet boundary conditions for B_2 then B_5 will necessarily be subject to Neumann boundary conditions.

The boundary condition $\langle B|$ is gapped / topological. All local degrees of freedom of the original QFT are localized to $|Z\rangle$. In contrast, $|Z\rangle$ is a so-called enriched Neumann boundary condition [3] or simply referred to as the physical boundary condition. See figure 1. In part, boundaries are distinguished by which of the above bulk operators can end on these. Only a maximal set of mutually local operators can end on $\langle B|$, while all operators can end on $|Z\rangle$.

Therefore, in our gauge theory example, the covector $\langle B|$ contains both the data of the chosen global form of the gauge group G as well as background values for the realized higher symmetries. The gauge group is equivalent to a choice of polarization. Defect operators are identified as operators

²To exemplify, consider the more familiar case of a 1-form gauge potential A_1 . When taking values in $U(1)$ its gauge transformations are characterized by some $g = \exp(i\lambda_0) \in U(1)$ via $A_1 \rightarrow A_1 \mapsto g^{-1}dg = A_1 + d\lambda_0$ and $\lambda_0(x) \sim \lambda_0(x) + 2\pi$ that is $\lambda(x) \in U(1)$. In contrast, when taking values in \mathbb{R} its gauge transformations are $A_1 \rightarrow A_1 + d\lambda_0(x)$ with $\lambda_0(x) \in \mathbb{R}$.

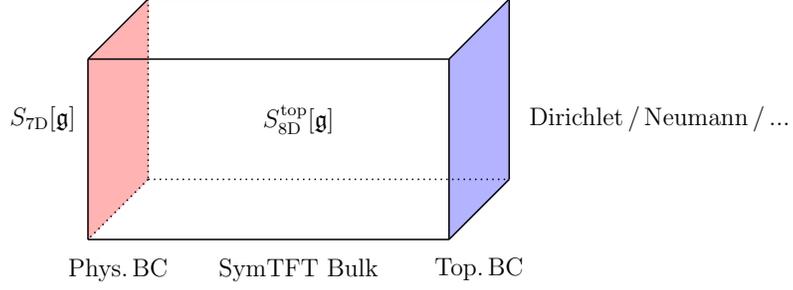


Figure 1: Sketch of the SymTFT slab $M \times I$. The physical boundary condition and SymTFT bulk are independent of the global form of the gauge group which is only determined by the topological boundary conditions.

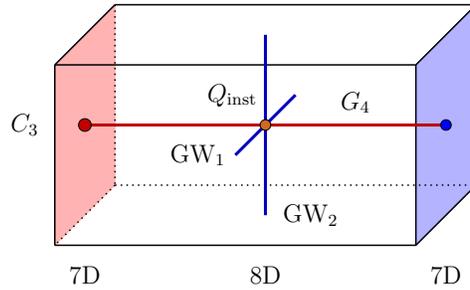


Figure 2: We sketch the configuration of two intersecting topological Gukov-Witten operators lifted to the SymTFT bulk.

which can terminate on both boundaries $\langle B |$ and $| Z \rangle$. Symmetry operators are operators which can not end on $\langle B |$. See figure 2.

To be Continued.

References

- [1] A. Kapustin and N. Seiberg, “Coupling a QFT to a TQFT and Duality,” *JHEP* **04** (2014) 001, [arXiv:1401.0740 \[hep-th\]](#).
- [2] S. Gukov, P.-S. Hsin, and D. Pei, “Generalized global symmetries of $T[M]$ theories. Part I,” *JHEP* **04** (2021) 232, [arXiv:2010.15890 \[hep-th\]](#).
- [3] J. Kaidi, K. Ohmori, and Y. Zheng, “Symmetry TFTs for Non-invertible Defects,” *Commun. Math. Phys.* **404** no. 2, (2023) 1021–1124, [arXiv:2209.11062 \[hep-th\]](#).