

Symmetry Theories, Wigner's Function, Compactification, and Holography



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Motivation

- Intrinsically Relative Theories (e.g., 6D SCFTs with $|\mathbb{D}^{(2)}| \neq N^2$)

[Witten; 1998], ..., [Gukov, Hsin, Pei; 2020]

Not always possible: Relative QFT $\rightarrow$ Absolute QFT

Partition Vector $\rightarrow$ Partition Function

- SymTFT Framework for Mixed States [Luo, Wang, Bi; 2025], [Schäfer-Nameki, Tiwari, Warman, Zhang; 2025], [Qi, Sohal, Chen, Stephen, Prem; 2025]

- a) String Compactifications: Mixed State Boundary conditions
- b) Holographic Systems: Ensemble of Theories

Outline

- Wigner's Quasi-Probabilistic Function
 - a) Continuum Case
 - b) Discrete Case
 - c) Wavefunction Interpretation
- Wigner's Quasi-Probabilistic Function and SymTFTs for Mixed States
 - a) Folding and Unfolding
 - b) Examples
- Compactification (and Holography)

Wigner (1932)

Classical Statistical Mechanics: Gibbs-Boltzmann formula

$$P(x, p) dx dp = e^{-\beta \epsilon} dx dp, \quad \epsilon = \frac{p^2}{2m} + V(x), \quad \beta = \frac{1}{k_B T}$$

Quantum Mechanics: von Neumann formula

$$\langle Q \rangle = \text{Tr } Q e^{-\beta H}$$

Drawback: challenging computationally.

Wigner (1932): Continuum Case

Wigner's Alternative: Start with phase space density

$$W(x, p) = \frac{1}{\pi} \int dy \psi(x - y) e^{2ipy} \psi(x + y)^*$$

with QM Wavefunction $\psi(x) = \langle x | \psi \rangle$ (pure state). [Wigner, 1932]

Integration over Lagrangian submanifolds in phase space

$$\int dp W(x, p) = |\psi(x)|^2 \quad \text{and} \quad \int dx W(x, p) = \left| \int dx \psi(x) e^{-ipx} \right|^2 = |\tilde{\psi}(p)|^2$$

Key Equation: Expectation values as phase space integrals

$$\int dx dp W(x, p) g(x, p) = \langle \hat{G} \rangle$$

Wigner (1932): Continuum Case

Generalization to Mixed States:

$$W_{\rho}(x, p) = \frac{1}{\pi} \int dq \langle x - q | \hat{\rho} | x + q \rangle e^{2ipq}, \quad \hat{\rho} = \sum p_i |\psi_i\rangle \langle \psi_i|$$

Integration over Lagrangian Submanifolds in Phase Space:

$$\int dp W_{\rho}(x, p) = \langle x | \hat{\rho} | x \rangle, \quad \int dx W_{\rho}(x, p) = \langle p | \hat{\rho} | p \rangle$$

Expectation Values as Phase Space Integrals:

$$\text{Tr } \hat{\rho} \hat{G} = \int dx dp W_{\rho}(x, p) g(x, p)$$

Wigner (1932): Continuum Case

Comments:

- $W_\rho(x, p)$ takes negative values $\rightarrow$ “Quasi-Probabilistic Function”
- Non-Uniqueness: Function of N variables $\mapsto$ Function of $2N$ variables

$$W_\rho^{[r,s]}(x, p) = \frac{r+s}{2\pi} \int dq \langle x - rq | \hat{\rho} | x + sq \rangle e^{(r+s)ipq}$$

- Computationally favorable: Explicit Quantum Corrections $f_k^{[r,s]}$

$$W_{\rho_t}^{[r,s]}(x, p) = e^{-\beta\epsilon} + \hbar f_1^{[r,s]} + \hbar^2 f_2^{[r,s]} + \dots$$

Discrete Case

Phase space $(x, p) \in \mathbb{Z}_N \times \mathbb{Z}_N$, then [Wootters; 1987], [Bouzouina, Bievre; 1996], [Bianucci, Miquel, Paz, Saraceno; 2001], ..., [Heckman, Hübner, Murdia; 2025]

$$W_{\rho}^{[r,s]}(x, p) = \frac{1}{2\pi} \sum_q \langle x - rq | \hat{\rho} | x + sq \rangle e^{(r+s)ipq}, \quad \gcd(r+s, N) = 1$$

with normalization factor change due to

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{\frac{2\pi i}{N} g n(n'-n'')} = \delta_{n', n''}$$

for all units of $g \in \mathbb{Z}_N$, as characterized by $\gcd(g, N) = 1$.

Wavefunction Interpretation

Introduce displacement, dilation and conjugation operators

$$\begin{aligned}
 U_x &= \exp(-ix\hat{p}) & \text{with} & & U_x |x'\rangle &= |x+x'\rangle \\
 V_p &= \exp(+ip\hat{x}) & \text{with} & & V_p |p'\rangle &= |p+p'\rangle \\
 \Delta_r & & \text{with} & & \Delta_r |x\rangle &= |rx\rangle \\
 \mathcal{C} & & \text{with} & & \mathcal{C} |x\rangle &= |-x\rangle, \mathcal{C} |p\rangle = |-p\rangle
 \end{aligned}$$

and the doubled operator

$$\begin{aligned}
 \mathbb{U}_{x,p} &\equiv \exp(-ix\hat{p} + ip\hat{x}) \\
 \mathbb{U}_{x,p}^{[r]} &\equiv \mathbb{U}_{x,p} \Delta_r
 \end{aligned}$$

Then, with phase cancellation due to Baker-Campbell-Hausdorff,

$$\begin{aligned}
 W_\rho^{[r,s]}(x,p) &= \frac{1}{2\pi} \sum_q \langle x - rq | \hat{\rho} | x + sq \rangle e^{(r+s)ipq} \\
 &= \frac{1}{2\pi} \text{Tr}(\hat{\rho} \mathbb{U}_{x,p}^{[s]} \mathcal{C} \mathbb{U}_{x,p}^{[r]\dagger}) = \frac{1}{2\pi} \sum_i p_i \langle Z_i | \mathbb{U}_{x,p}^{[s]} \mathcal{C} \mathbb{U}_{x,p}^{[r]\dagger} | Z_i \rangle
 \end{aligned}$$

Wavefunction Interpretation

Given the N -dimensional Hilbert space $\mathcal{H}$ consider the double

$$\mathcal{H} \otimes \overline{\mathcal{H}}$$

and map the mixed state $\hat{\rho}$ to the vector (operator-state mapping)

$$|\rho\rangle\rangle \equiv \sum_i p_i |i\rangle \otimes |\bar{i}\rangle$$

and, with respect to the point $\Phi = (x, p)$ in phase space, define

$$\langle\langle \Phi^{[r,s]} | \equiv \sum_e \langle e | \mathcal{C} \mathbb{U}_{\Phi}^{[r]}{}^\dagger \otimes \langle e | \mathbb{U}_{\Phi}^{[s]}{}^\dagger$$

which derives from the Choi state [\[Choi; 1975\]](#)

$$|\text{Choi}\rangle \propto \sum_e |e\rangle \otimes |\bar{e}\rangle$$

Wavefunction Interpretation

Previous definitions are such that

$$\langle\langle \Phi^{[r,s]} | \rho \rangle\rangle = W_{\rho}^{[r,s]}(\Phi)$$

with phase space point $\Phi = (x, p)$. Further, given a classical function $f(\Phi)$ on phase space define

$$\langle\langle f^{[r,s]} | \equiv \sum_{\Phi} f(\Phi) \langle\langle \Phi^{[r,s]} |$$

then one has the overlap

$$\langle\langle f^{[r,s]} | \rho \rangle\rangle = \sum_{\Phi} f(\Phi) \langle\langle \Phi^{[r,s]} | \rho \rangle\rangle = \sum_{\Phi} f(\Phi) W_{\rho}^{[r,s]}(\Phi).$$

SymTFT and Wavefunction Interpretation

Consider a relative QFT with partition vector $|Z\rangle$ and a choice of absolute form (assuming existence) as specified by the Lagrangian

$$\Lambda \subset \mathbb{D}$$

specifying electric and magnetic bases to the SymTFT Hilbert space $\mathcal{H}$. Then, the QFT partition function with some background $e \in \Lambda$ is

$$Z(\Lambda, e) \equiv \langle \Lambda, e | Z \rangle .$$

[Reshetikhin, Turaev; 1991], [Turaev,Viro;1992], [Barrett, Westbury; 1996], [Witten; 1998], [Fuchs, Runkel, Schweigert; 2002], [Kapustin, Saulina; 2010], [Kitaev, Kong; 2011], [Freed, Teleman; 2014], [Gaiotto, Kapustin, Seiberg, Willett; 2014], [Gaiotto and J. Kulp; 2021], [Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki; 2021], [Freed, Moore, Teleman; 2022], [Kaidi, Ohmori, Zheng; 2022], ...

SymTFT and Wavefunction Interpretation

Independent of a choice of polarization we can associate Wigner's function to a pure state

$$\hat{\rho}_Z = |Z\rangle\langle Z| \mapsto W_Z^{[r,s]}(e, m),$$

and from it one extracts the partition functions squared

$$\sum_{m \in \bar{\Lambda}} W_Z^{[r,s]}(e, m) = |Z(\Lambda, e)|^2, \quad \sum_{e \in \Lambda} W_Z^{[r,s]}(e, m) = |Z(\bar{\Lambda}, m)|^2,$$

summing over Lagrangian submanifolds in phase space.

Mixed States and SymTFTs

More generally consider the mixed states

$$\hat{\rho}_Z = \sum_i z_i |Z_i\rangle\langle Z_i|, \quad \hat{\rho}_B = \sum_i b_i |B_i\rangle\langle B_i|$$

and define on phase space the classical function $p_B^{[r,s]}(\Phi)$ by

$$\mathrm{Tr}(\hat{\rho}_B \hat{\rho}_Z) \equiv \sum_{\Phi} p_B^{[r,s]}(\Phi) W_{\rho_Z}^{[r,s]}(\Phi).$$

Then define with respect to the doubled SymTFT Hilbert space $\mathcal{H} \otimes \overline{\mathcal{H}}$

$$\langle\langle \rho_B | \equiv \sum_{\mathbb{B}} p_B^{[r,s]}(\Phi) \langle\langle \Phi^{[r,s]} |$$

which is such that

$$\langle\langle \rho_B | \rho_Z \rangle\rangle = \mathrm{Tr}(\hat{\rho}_B \hat{\rho}_Z)$$

Mixed States and SymTFTs

Overall: SymTFT with Hilbert space $\mathcal{H} \otimes \overline{\mathcal{H}}$ and boundary conditions

$$\langle\langle \rho_B | \quad \text{and} \quad | \rho_Z \rangle\rangle$$

derived from mixed states of the original SymTFT with Hilbert space $\mathcal{H}$.



The topological boundary condition $\langle\langle \rho_B |$ is defined with respect to any function on phase space, independent of the existence of a polarization Λ .

A basis to the space of boundary conditions, for any fixed $[r, s]$, are

$$\langle\langle \Phi^{[r,s]} | = \sum_e \langle e | \mathcal{C} \mathbb{U}_\Phi^{[r]}{}^\dagger \otimes \langle e | \mathbb{U}_\Phi^{[s]}{}^\dagger$$

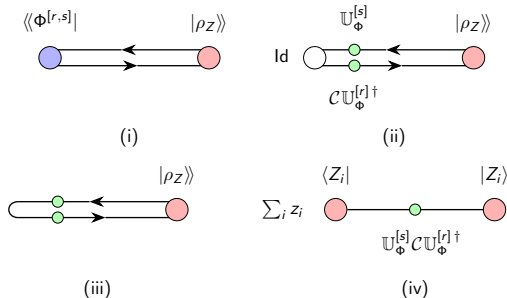
labelled by points Φ in the phase space of the original SymTFT.

Doubled SymTFT Sandwich: $W_{\rho_Z}^{[r,s]}(\Phi)$ in Pictures

Evaluating the trace:

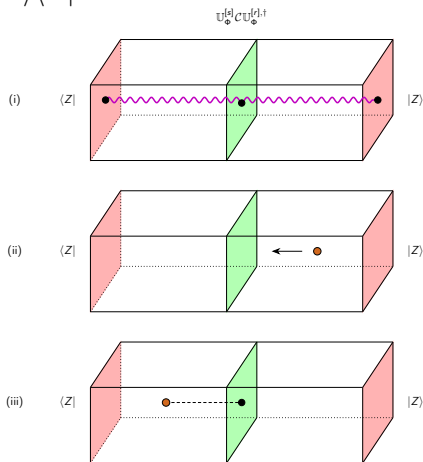
$$W_{\rho_Z}^{[r,s]}(\Phi) = \langle\langle \Phi^{[r,s]} | \rho_Z \rangle\rangle = \sum_i z_i \langle Z_i | \mathbb{U}_\Phi^{[s]} \mathcal{C} \mathbb{U}_\Phi^{[r]\dagger} | Z_i \rangle$$

Pictorial Presentation:



Defect and Symmetry Operators: Standard Story

Pure state $\hat{\rho} = |Z\rangle\langle Z|$:



Example: 6D SCFTs

3-form potential C^I and level matrix K_{IJ} (ADE Cartan Matrix):

$$\mathcal{S}_{7D} = \frac{K_{IJ}}{4\pi} \int_{7D} C^I dC^J,$$

Group of Surface Defects: $\mathbb{D} \cong \text{Coker}(K_{IJ})$.

$$|\mathbb{D}| \neq N^2 \rightarrow \text{Intrinsically Relative}$$

Topological bulk operators $\mathcal{T}_\nu(\Sigma) = \exp\left(i \int_\Sigma \nu_I \hat{C}^I\right)$ and linking

$$\mathcal{T}_\nu(\Sigma) \mathcal{T}_{\nu'}(\Sigma') = \exp\left(2\pi i \times \nu_I K^{IJ} \nu'_J \times \text{lnk}(\Sigma, \Sigma')\right) \mathcal{T}_{\nu'}(\Sigma') \mathcal{T}_\nu(\Sigma)$$

[Monnier; 2018], [Heckman, Tizzano; 2018], [Gukov, Pei, Putrov, Vafa; 2018], [Hsieh, Tachikawa, Yonekura; 2020],

[Gukov, Hsin, Pei; 2020], ...

Example: 6D SCFTs

Defect group $\mathbb{D}$ is the “phase space” and

$$\mathbb{U}_\nu = \exp \left(-i K_{IJ} \int_{6D} C^I \hat{C}^J \right)$$

with label $[\nu] \in \mathbb{D}$ where $\nu_J = - \int_{3D} K_{IJ} C^I$.

E.g., for the pure state $\hat{\rho}_Z = |Z\rangle\langle Z|$ we have Wigner's function

$$W_Z(\nu) = \langle\langle \nu | \rho_Z \rangle\rangle = \langle Z | \mathbb{U}_\nu \mathcal{C} \mathbb{U}_\nu^\dagger | Z \rangle ,$$

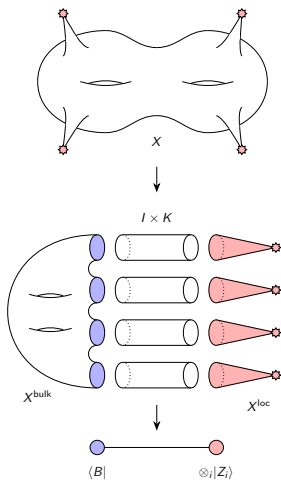
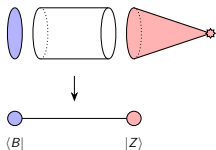
where $r = s = 1$. This is the partition function of a doubled system of 6D SCFTs (which is absolute) with 2-form symmetry group

$$\Gamma^{(2\text{-form})} \cong \mathbb{D}^\vee .$$

Compactification

Geometric Engineering:

$$X \xrightarrow[\text{\& Local Limit}]{\text{IIA/IIB/M/F}} \mathcal{T}_X$$



[Cvetič, Heckman, Hübner, Torres; 2022], [Gould, Lin, Sabag; 2023], [Cvetič, Dierig, Lin, Torres, Zhang; 2024]

Compactification: The Quiche

Decomposition of X with cross section K :

$$X = X^o \cup (I \times K) \cup X^{\text{loc}}$$

where $X^{\text{loc}} = \sqcup_i X_i^{\text{loc}}$ is a disjoint union of local models centered on all loci supporting localized degrees of freedom.

Symmetry theory derives from the cross section:

$$K \mapsto \mathcal{S} = \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_{\pi_0(K)}$$

Similarly, we have pure state physical boundary conditions:

$$X^{\text{loc}} \mapsto |Z\rangle\rangle = |Z_1\rangle \otimes \cdots \otimes |Z_{\pi_0(X^{\text{loc}})}\rangle$$

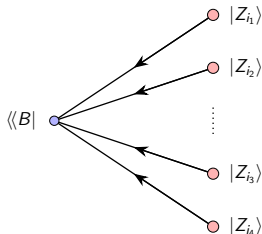
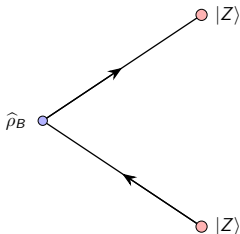
[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki; 2021], [Heckman, MH, Torres, Turner, Yu; 2023]

Compactification: “Mixed State Boundary Conditions”

In contrast, boundary conditions do not decompose into tensor products:

$$X^\circ \mapsto |B\rangle\rangle \neq \otimes_i |B_i\rangle$$

Compare to Wigner's function (left) approach:



Compactification: Mixed State Boundary Conditions

Decompose (assuming existence of the B_i)

$$|B\rangle\rangle = \sum_{i_1, \dots, i_m} |B_{i_1, \dots, i_m}\rangle\rangle \langle\langle B_{i_1, \dots, i_m} | B \rangle\rangle \equiv \sum_{i_1, \dots, i_m} b_{i_1, \dots, i_m}^* |B_{i_1, \dots, i_m}\rangle\rangle ,$$

where $m = \pi_0(K)$ is the number of connected components of K and

$$|B_{i_1, \dots, i_m}\rangle\rangle = |B_{i_1}\rangle \otimes \dots \otimes |B_{i_m}\rangle .$$

Coefficients $b_{i_1, \dots, i_m} = \langle\langle B_{i_1, \dots, i_m} | B \rangle\rangle$ are functions of the background fields.

The overall partition function is

$$\langle\langle B | Z \rangle\rangle = \sum_{i_1, \dots, i_m} b_{i_1, \dots, i_m} \langle B_{i_1} | Z_{i_1} \rangle \dots \langle B_{i_m} | Z_{i_m} \rangle ,$$

which is a collection of direct products of absolute theories with partition functions $\langle B_{i_k} | Z_{i_k} \rangle = Z_{i_k}(B_{i_k})$ weighted by $b_{i_1, \dots, i_m}$.

Example: M-theory on $T^6/\mathbb{Z}_3$

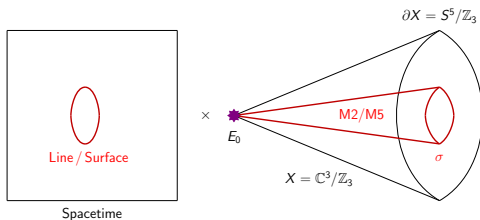
$X = T^6/\mathbb{Z}_3$ contains 27 singularities modelled on $\mathbb{C}^3/\mathbb{Z}_3$

$\Rightarrow$ M-theory on $X = T^6/\mathbb{Z}_3$ contains 27 Seiberg SCFTs $E_0^{\oplus 27}$

Each Seiberg SCFT E_0 has defect lines and surfaces

$$\mathbb{D}^{(1)} \cong \mathbb{Z}_3^{(M2)}, \quad \mathbb{D}^{(2)} \cong \mathbb{Z}_3^{(M5)}$$

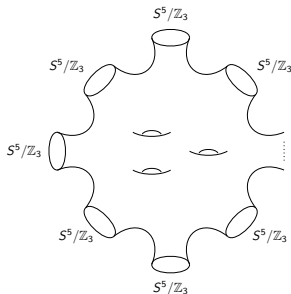
from branes on cones over $H_1(S^5/\mathbb{Z}_3) \cong \mathbb{Z}_3$ and $H_3(S^5/\mathbb{Z}_3) \cong \mathbb{Z}_3$.



[Del Zotto, Heckman, Park, Rudelius; 2015], [Morrison, Schäfer-Nameki, Willett; 2020], [Cvetič, Heckman, Hübner,

Example: M-theory on $T^6/\mathbb{Z}_3$

The 27 copies of the 5D SCFT E_0 interact across the bulk X° :



$$H_n(T^6/\mathbb{Z}_3) = \begin{cases} \mathbb{Z} & n = 6 \\ 0 & n = 5 \\ \mathbb{Z}^9 \oplus \mathbb{Z}_3^4 & n = 4 \\ \mathbb{Z}^2 & n = 3 \\ \mathbb{Z}^9 \oplus \mathbb{Z}_3^{17} & n = 2 \\ 0 & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

Boundary conditions $\langle\langle B |$ compute from $H_n(T^6/\mathbb{Z}_3)$. For example, an $\mathfrak{u}(1)^{\oplus 9}$ gauge theory (abelian bulk modes) is associated to $\langle\langle B |$ and determines the coefficients $b_{i_1, \dots, i_m}$. [Cvetič, Heckman, Hübner, Torres; 2022]

Computational Details for $\langle\langle B|$

Notation:

$$X = T^6/\mathbb{Z}_3, X^{\text{loc}} = \sqcup_{i=1}^{27} (\mathbb{C}^3/\mathbb{Z}_3)_i, \partial X^{\text{loc}} = \sqcup_{i=1}^{27} (S^5/\mathbb{Z}_3)_i, X^\circ = X \setminus X^{\text{loc}}$$

Sequences setting boundary conditions:

$$0 \rightarrow H_4(X^\circ) \xrightarrow{j_4} H_4(X) \xrightarrow{\partial_4} H_3(\partial X^{\text{loc}}) \xrightarrow{i_3} \text{Tor } H_3(X^\circ) \rightarrow 0$$

$$0 \rightarrow \mathbb{Z}^9 \xrightarrow{j_4} \mathbb{Z}^9 \oplus \mathbb{Z}_3^4 \xrightarrow{\partial_4} \mathbb{Z}_3^{27} \xrightarrow{i_3} \mathbb{Z}_3^{17} \rightarrow 0$$

$$0 \rightarrow H_2(X^\circ) \xrightarrow{j_2} H_2(X) \xrightarrow{\partial_2} H_1(\partial X^{\text{loc}}) \xrightarrow{i_1} \text{Tor } H_1(X^\circ) \rightarrow 0$$

$$0 \rightarrow \mathbb{Z}^9 \xrightarrow{j_2} \mathbb{Z}^9 \oplus \mathbb{Z}_3^{17} \xrightarrow{\partial_2} \mathbb{Z}_3^{27} \xrightarrow{i_1} \mathbb{Z}_3^4 \rightarrow 0.$$

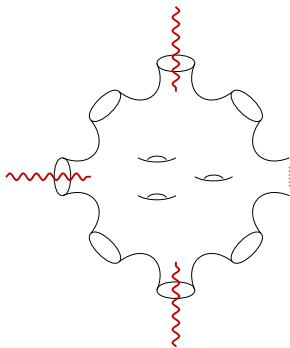
Boundary conditions are such that the overall p -form gauge groups $G^{(p)}$ are

$$G^{(0)} = (\mathbb{Z}_3^{23} \times U(1)^9) / \mathbb{Z}_3^6,$$

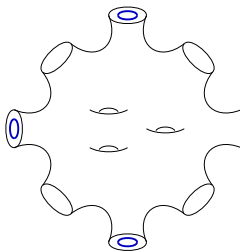
$$G^{(1)} = (\mathbb{Z}_3^{10} \times U(1)^9) / \mathbb{Z}_3^6,$$

$$G^{(2)} = U(1)^2.$$

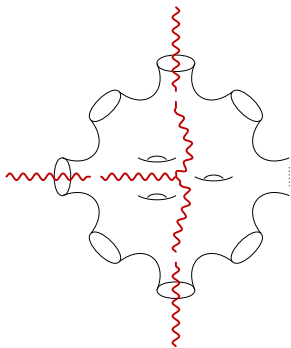
Defect Operators



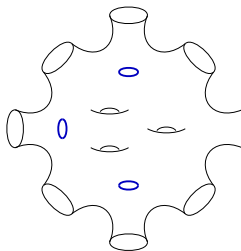
Symmetry Operators



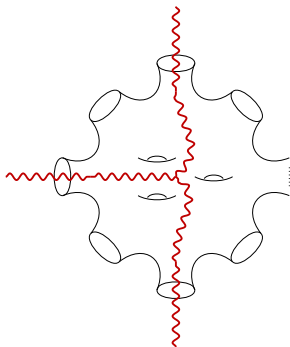
Defect Operators



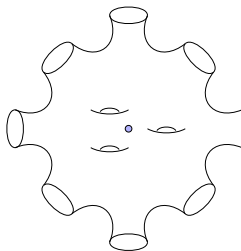
Symmetry Operators



Defect Operators

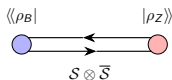


Symmetry Operators



Conclusion and Outlook

- Wigner's Quasi-Probabilistic Function applied to SymTFTs:
 - a) Pure states: Computable for intrinsically relative theories,
Independent of Lagrangian $\Lambda \subset \mathbb{D}$
 - b) Mixed states: SymTFT Sandwich



Wave- / Partition function interpretation: $W_{\rho_Z}^{[r,s]}(\Phi) = \langle\langle \Phi^{[r,s]} | \rho_Z \rangle\rangle$

- Applications: String Compactifications

$$\text{Bulk } X^\circ \mapsto \langle\langle X^\circ |$$

- Outlook: Holographic Ensembles