# On Frozen Singularities

## WIP w/ Lin, Torres, Xu

**Max Hübner - Tuesday July 11th, Beijing - Generalized Symmetries in HEP and CMP**

# Frozen Singularities?

## Example: ADE-Singularities

- M-theory Background: Geometry + Fractional 4-form Flux

$$M^{11} = \mathbb{R}^{1,6} \times X^4_{n/d} \qquad X^4_{n/d} = \mathbb{C}^2/\Gamma \quad \text{with} \quad \int_{\mathbb{C}^2/\Gamma} G_4 = \int_{S^3/\Gamma} C_3 = \frac{n}{d} \mod 1$$

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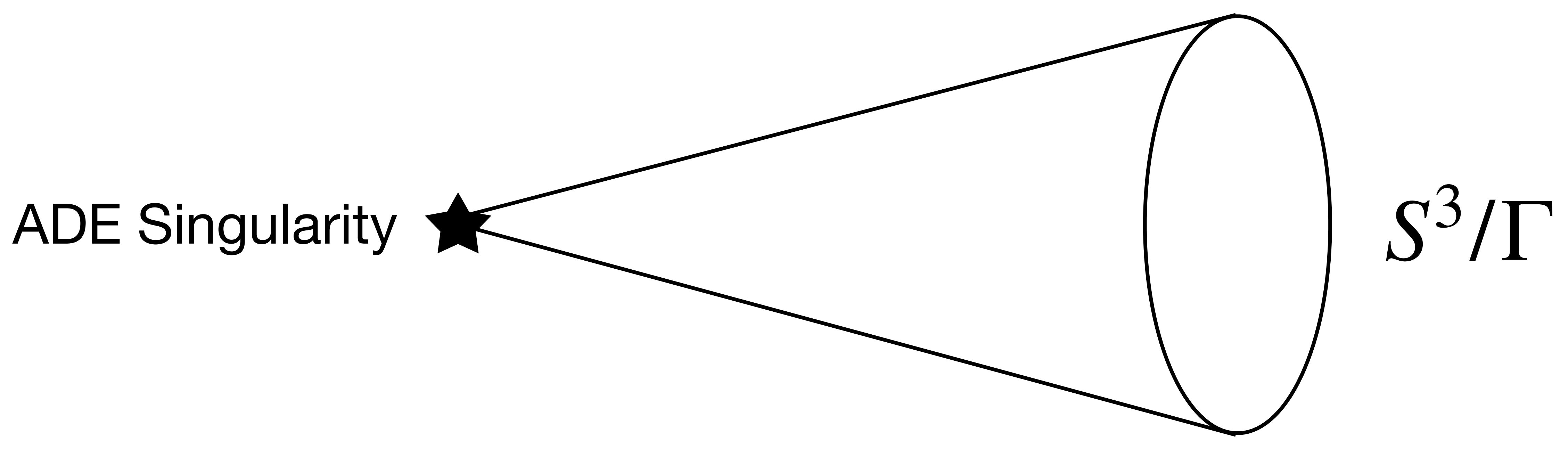


(unfrozen)

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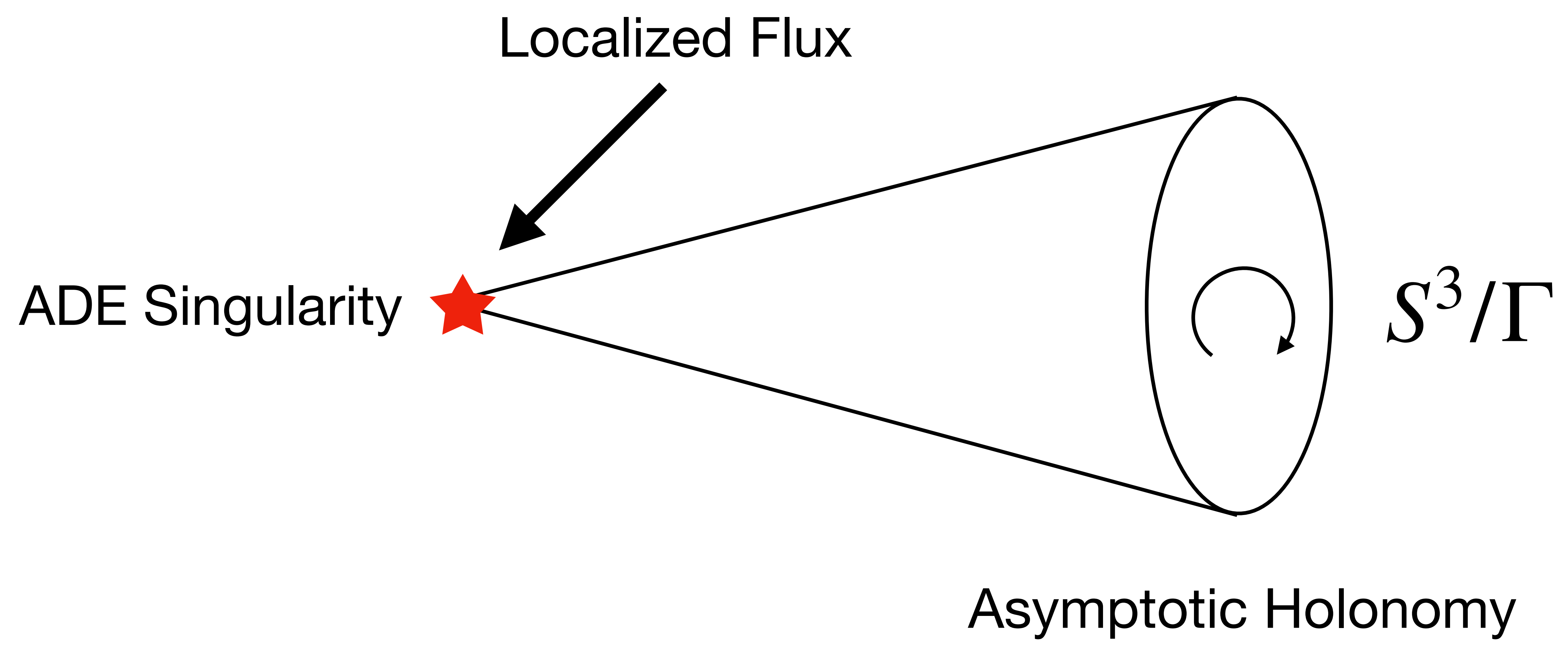


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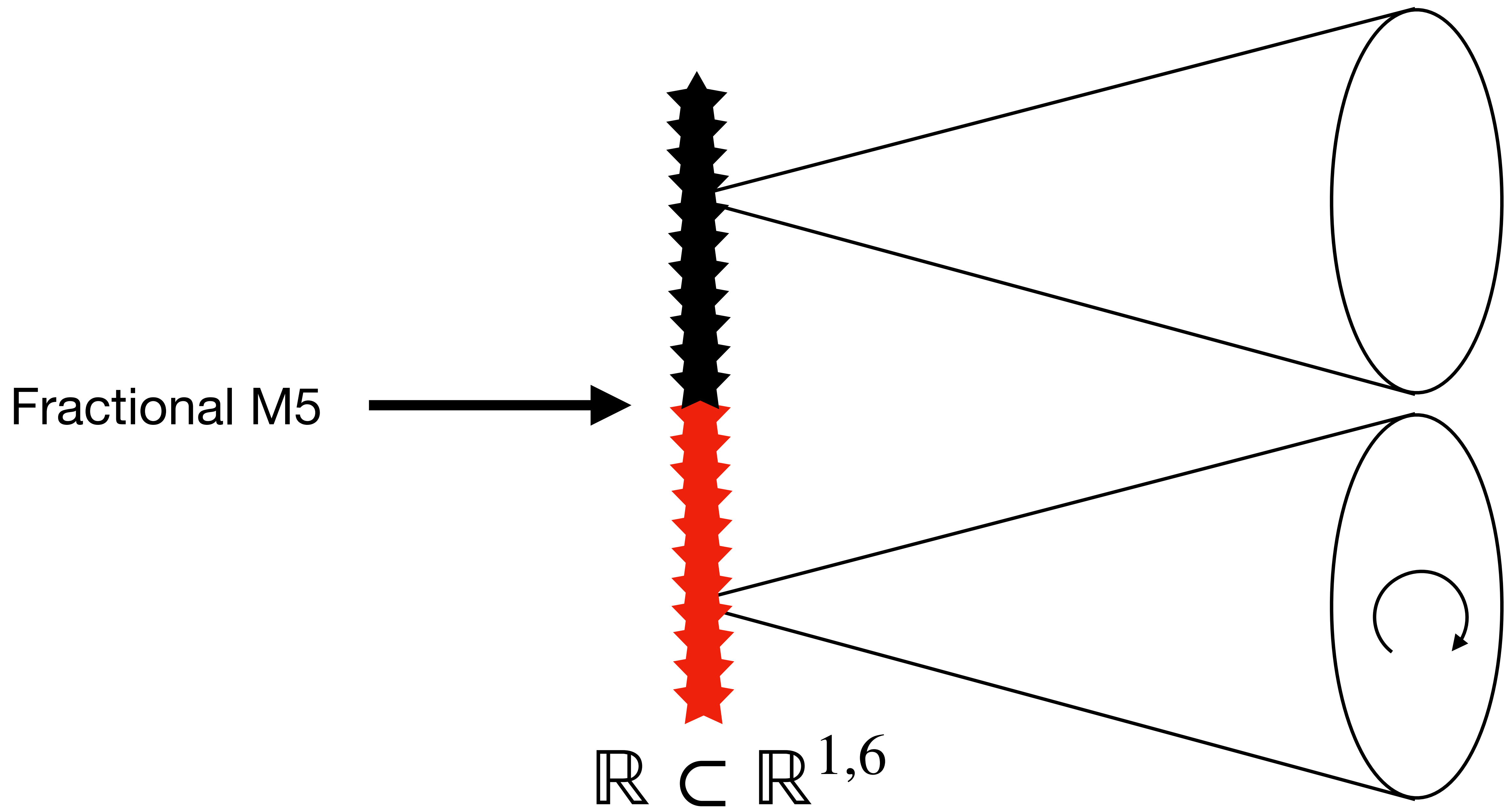


$$\int_{S^3/\Gamma} C_3 = 0 \mod 1$$

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- Geometric Engineering:

  1) ADE Singularities $\Rightarrow$ ADE Gauge Algebras. What about non-simply laced algebras?

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- Fractional Non-Perturbative Objects
- Fractional Fluxes and Confinement
- Generalizations $\longrightarrow$ Other Singularities, Compact Models,
  Stabilization of Non-SUSY Backgrounds, …

# Outline

1. Two Take Home Messages
2. Review
3. Frozen IIA ADE Singularities:

$$a) \quad \int_{\gamma_1 \subset \partial X} C_1^{RR} = \frac{s}{t} \qquad b) \quad \int_{\partial X} C_3^{RR} = \frac{n}{d}$$

4. Fractional M5-branes and K-theory
5. Generalizations and Conclusions

# Two Take Home Messages

IIA Background: Geometry + Fractional 4-form RR-Flux

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- D-brane Charges take values in K-theory: $K^0_{orb}(X)$

NS-Backrounds: $K^0_{orb,H}(X)$

RR-Backrounds: ???

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  RR-Backrounds: Subgroup $\subset K^0_{orb}(X)$

# Review

## What motivated the discovery of Frozen singularities?

- Global Models: String Compactifications with 16 supercharges contain disconnected rank reduced components (e.g., CHL vacua and heterotic/type I compactifications). These were missing in other (M-theory/Heterotic) duality frames. [Witten; 1998] [Bershadsky, Pantev, Sadov; 1999] [de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi; 2002]

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- Local Models: In IIB the O7+ Plane has the same SL(2,Z) monodromy and RR-brane charge as 8 D7-branes + O7- Plane. Discrete Data was missing in F-theory. [Tachikawa; 2016]

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- Bottom Up: Classifications of 6D SCFTs/LSTs initially resulted in models with no known F-theory construction. Frozen O7+ divisors supplied the missing ingredient. [Bhardwaj, Morrison, Tachikawa, Tomasiello; 2017] [Bhardwaj; 2020]

# Review

## What is known about Frozen Singularities?

[de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi; 2002] [Cvetic, Dierigl, Lin, Torres, Zhang; 2025]

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| $\mathfrak{h}_{n/d}$ | $\mathfrak{sp}_k$ | $\mathfrak{su}_3$ | $\varnothing$ | $\mathfrak{so}_7$ | $\mathfrak{su}_2$ | $\varnothing$ | $\mathfrak{f}_4$ | $\mathfrak{g}_2$ | $\mathfrak{su}_2$ | $\varnothing$ | $\varnothing$ |

Where $\dfrac{n}{d}$ is a 3D Chern-Simons Invariant and d a dimension of an irrep of $\Gamma$

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$$\Leftrightarrow$$

- Heterotic Background: Geometry + Holonomies

$$M^{10} = \mathbb{R}^{1,6} \times T^3 \qquad H = (H_1, H_2, H_3)\,, \qquad H_i = \int_{S_i^1} A_{E8\times E8}\,, \qquad [H_j, H_k] = 0$$

(but not simultaneously diagonalizable)

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## What is known about Frozen Singularities?

[Del Zotto, Heckman, Tomasiello, Vafa; 2015]

- M-theory Background: Geometry + M5-brane Probe

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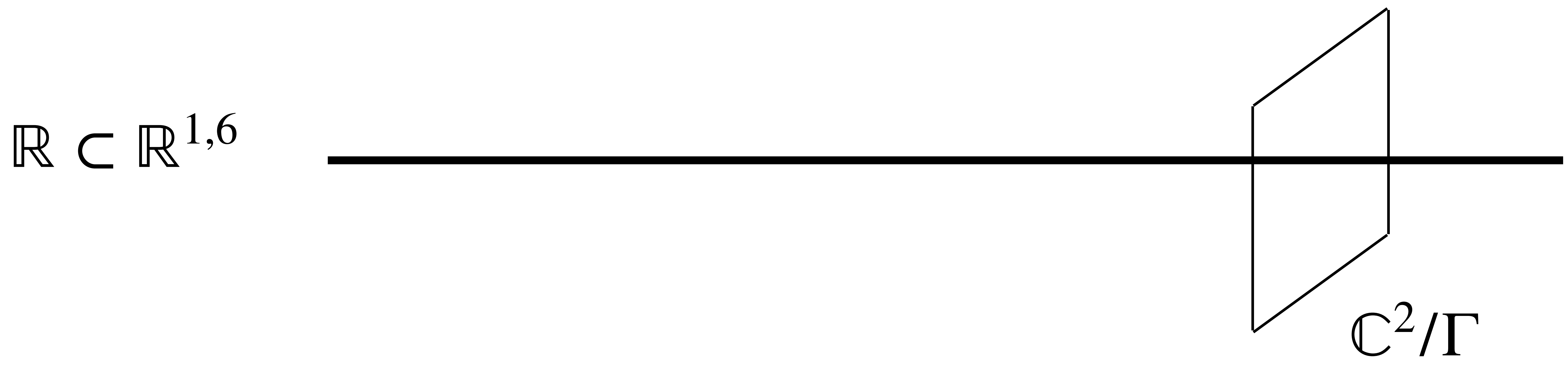

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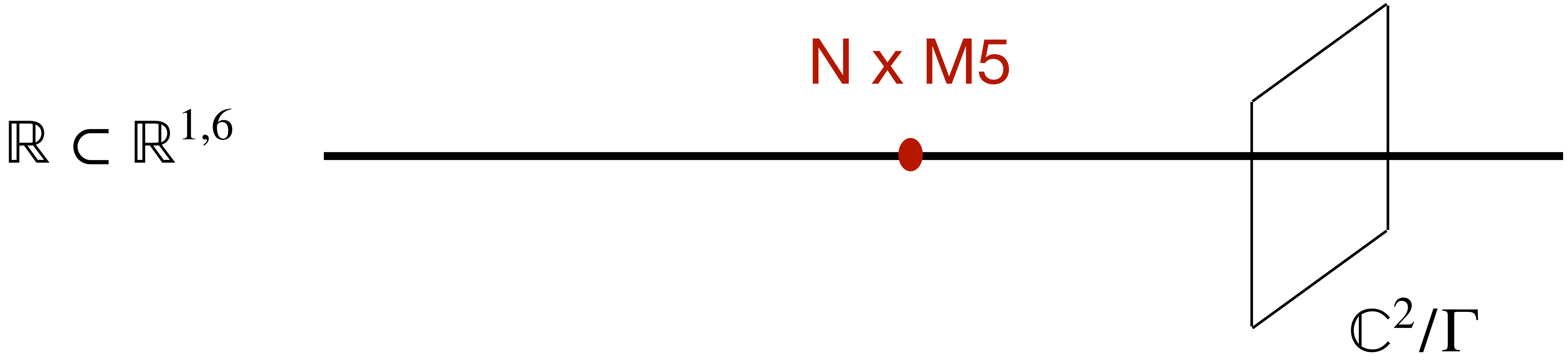

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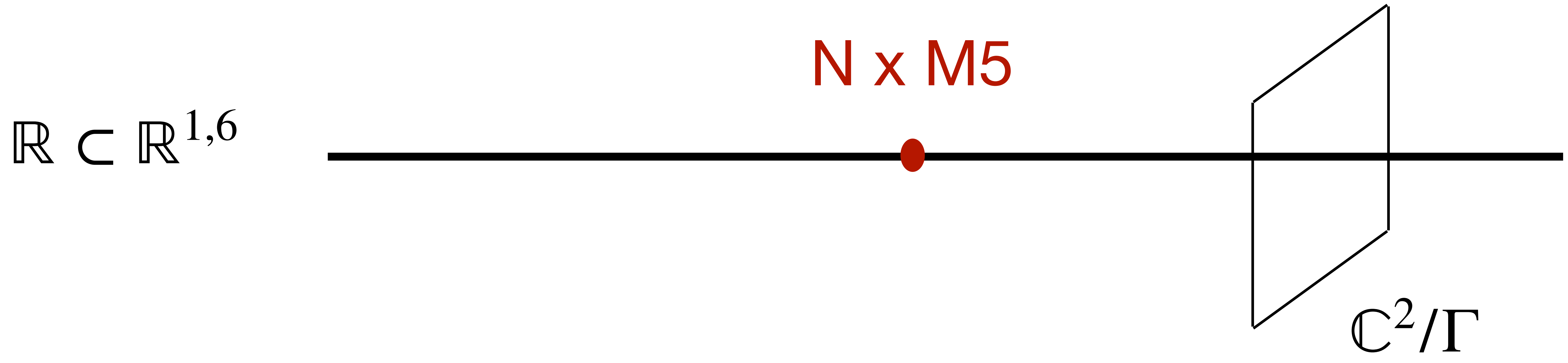


$\Leftrightarrow$

- F-theory Background: Geometry + Non-Minimal Singularities

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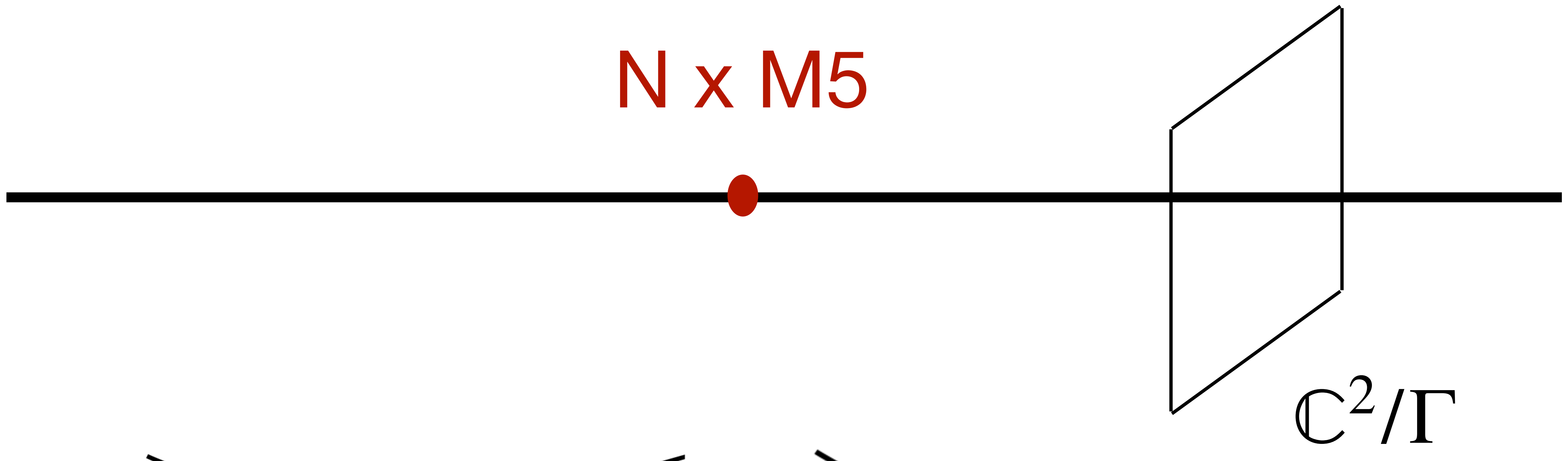


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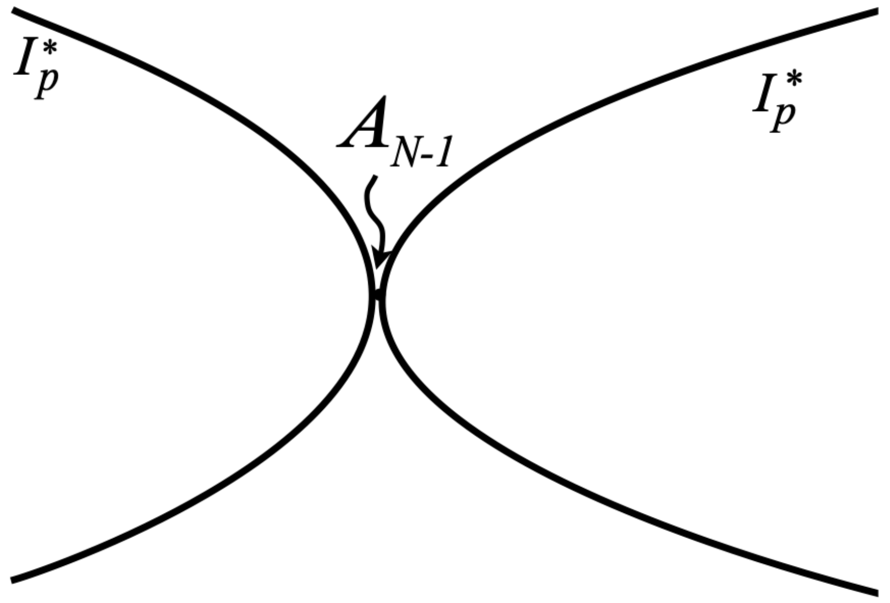
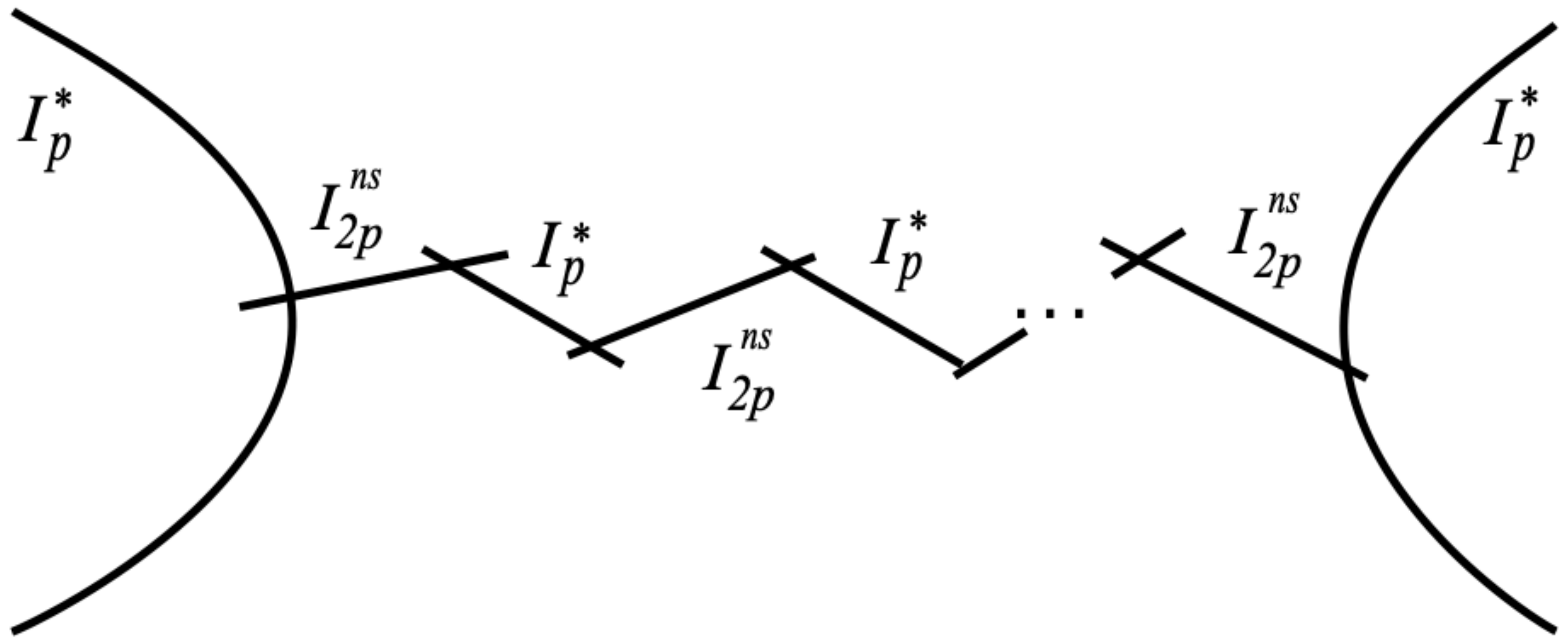
- F-theory Background:



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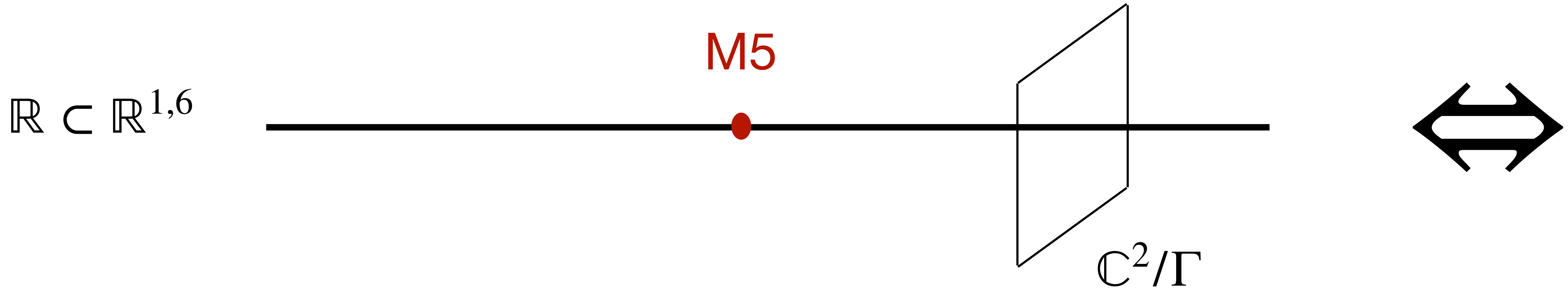

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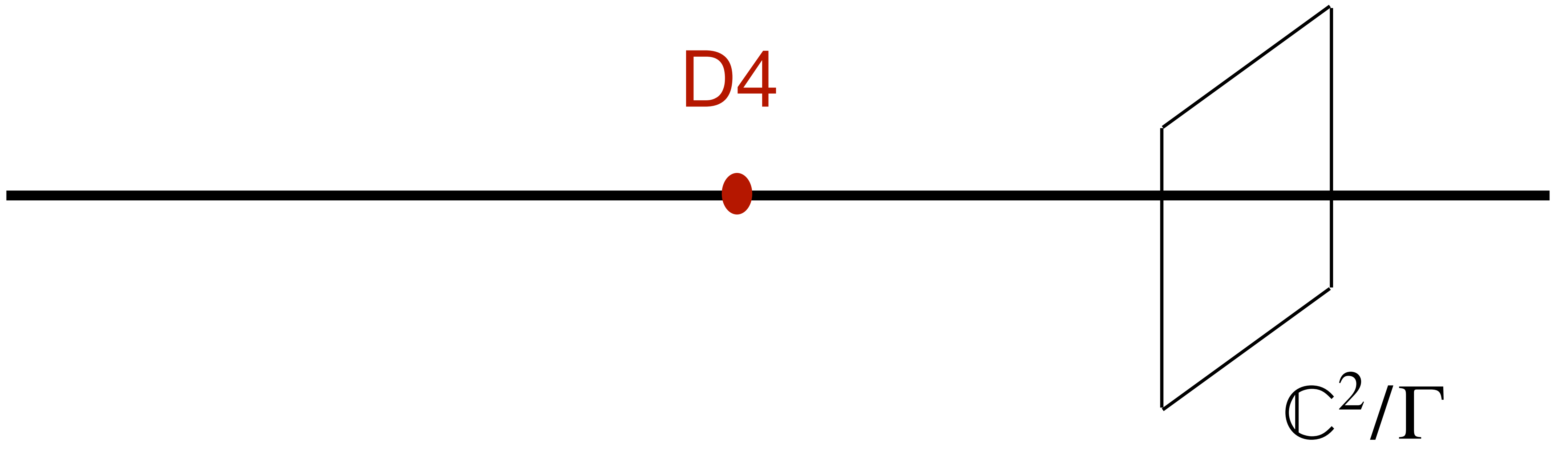


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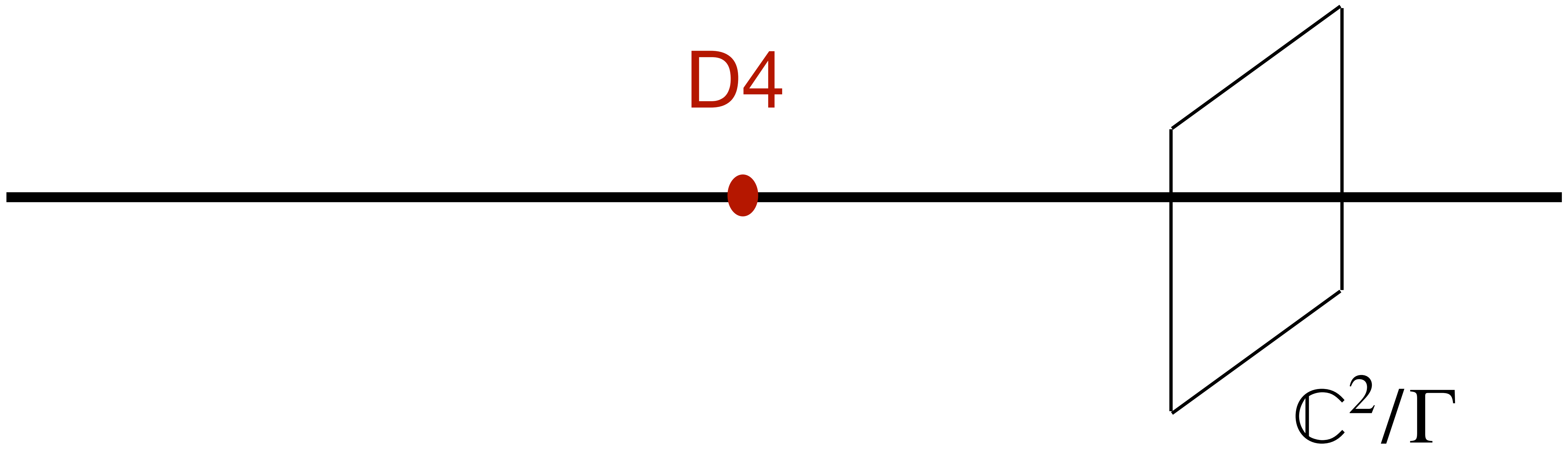


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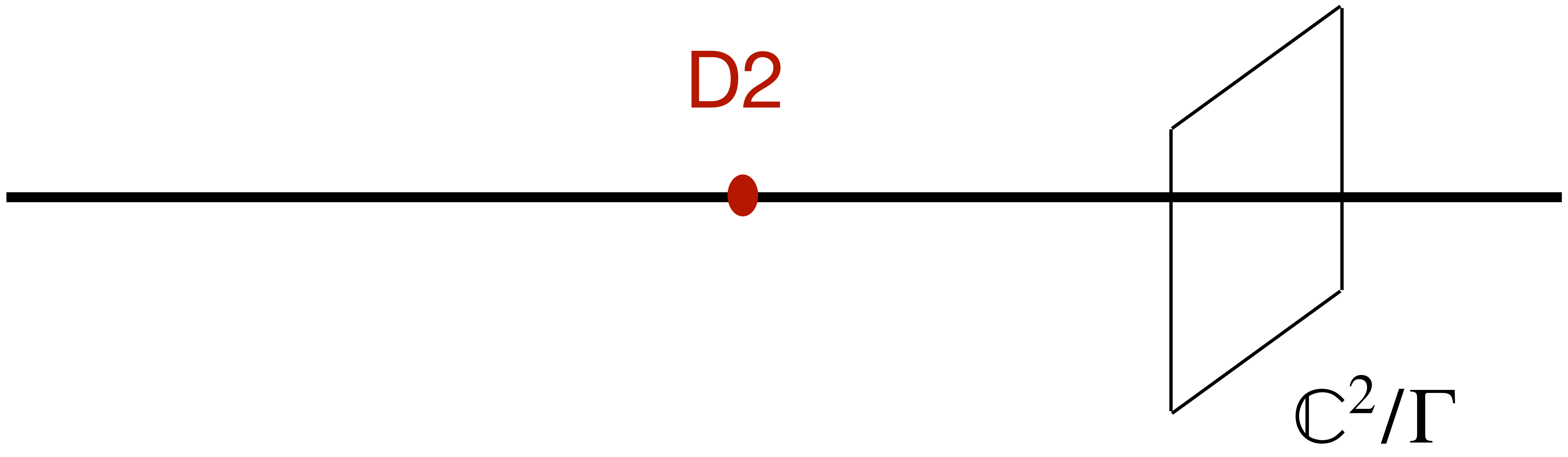


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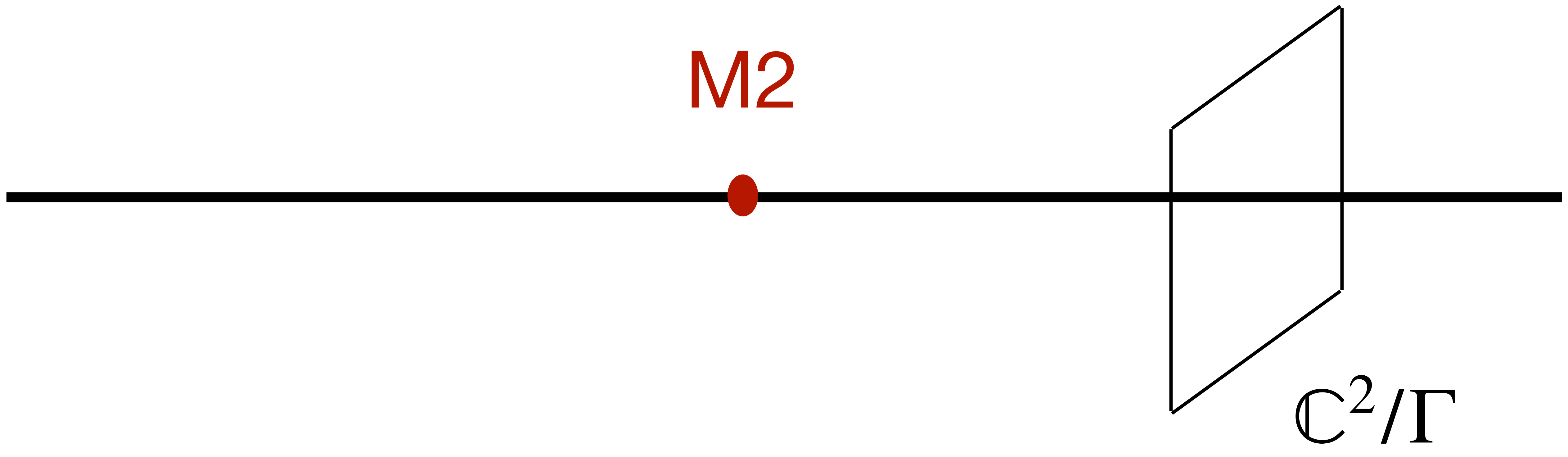


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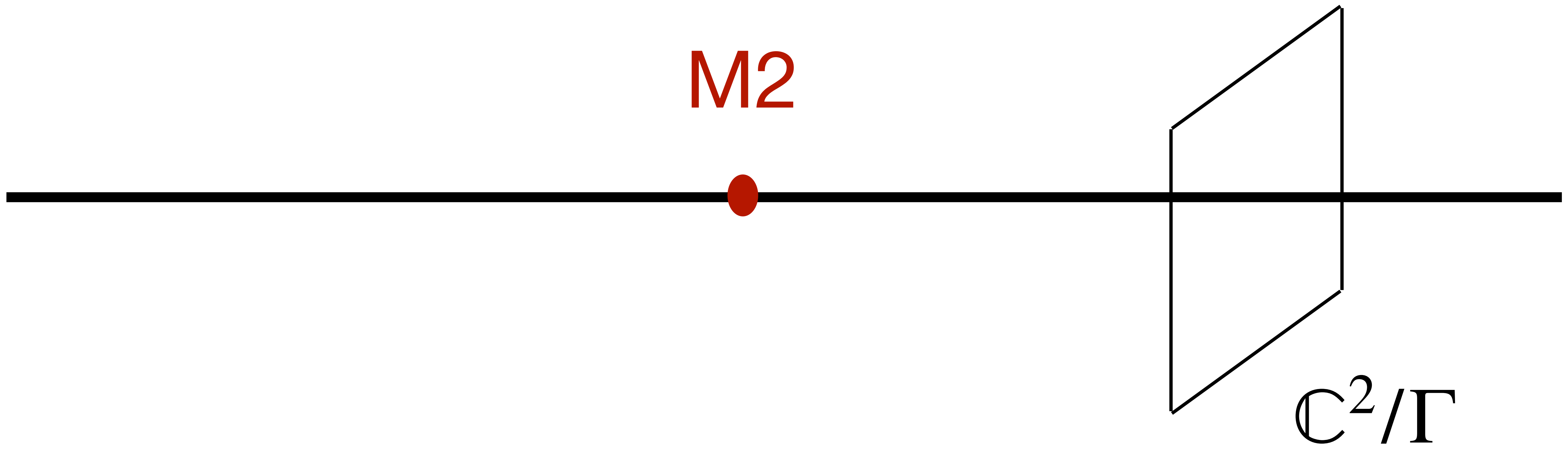


$\Leftrightarrow$

- M2 dissolves into the 7D SYM gauge theory as an instanton
- Leverage gauge theory description: Instantons can fractionate

**Note:**
**Changes in Duality frame to F-theory / Heterotic**
**or use of the ADE gauge theory**

**How to analyze the IIA / M-theory background directly?**

# Warm Up in IIA

# Warum Up

## … in IIA

- IIA Background: Geometry + Fractional 2-form RR-Flux

$$M^{10} = \mathbb{R}^{1,5} \times X^4_{s/t} \qquad X^4_{s/t} = \mathbb{C}^2/\mathbb{Z}_N \quad \text{with} \quad \int_{\mathbb{C}/\mathbb{Z}_N} F_2^{RR} = \int_{S^1/\mathbb{Z}_N} C_1^{RR} = \frac{s}{t} \mod 1$$

where t divides N and (s,t)=1.

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where t divides N and (s,t)=1.

- Why easy? We have a purely geometric M-theory frame:

$$M^{11} = \mathbb{R}^{1,5} \times (\mathbb{C}^2 \times S^1)/\mathbb{Z}_N = \mathbb{R}^{1,5} \times (\mathbb{C}^2/\mathbb{Z}_{N/t} \times S^1)/\mathbb{Z}_t$$

Geometry is ``less” singular. Moduli of $\mathbb{C}^2/\mathbb{Z}_N$ have been ``frozen”.

# Warum Up

## 2 representative cases

**IIA**

$$\int_{\mathbb{C}/\mathbb{Z}_2} F_2^{RR} = \int_{S^1/\mathbb{Z}_2} C_1^{RR} = \frac{1}{2} \mod 1$$

$$\int_{\mathbb{C}/\mathbb{Z}_4} F_2^{RR} = \int_{S^1/\mathbb{Z}_4} C_1^{RR} = \frac{1}{2} \mod 1$$

**M**

$(\mathbb{C}^2 \times S^1)/\mathbb{Z}_2$ (Smooth)

$(\mathbb{C}^2/\mathbb{Z}_2 \times S^1)/\mathbb{Z}_2$ $A_1 \subset A_3$

# Warum Up

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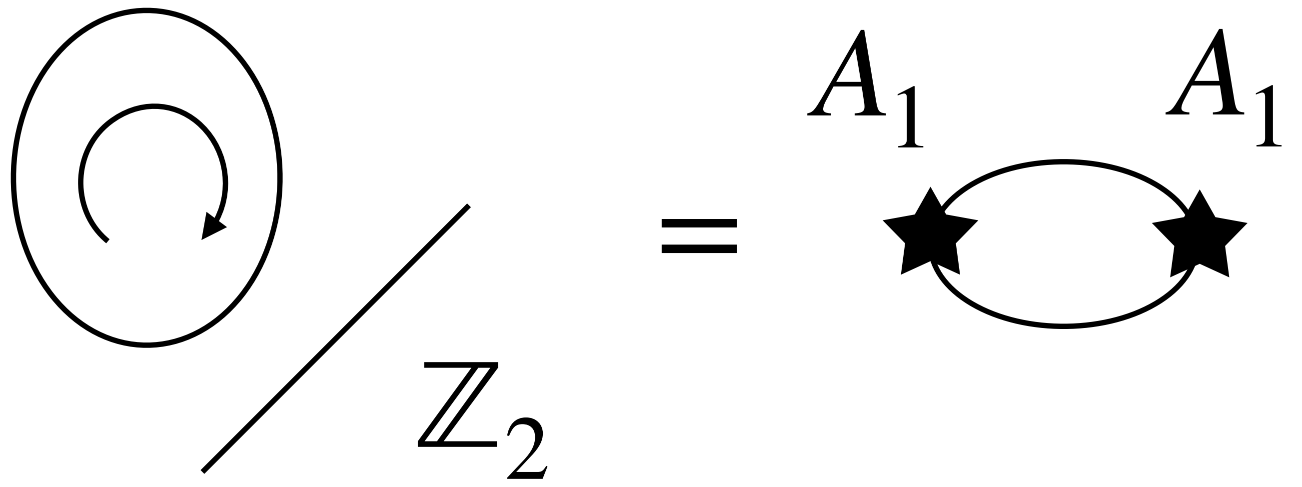


**M**

$$(\mathbb{C}^2 \times S^1)/\mathbb{Z}_2$$ (Smooth)

$$(\mathbb{C}^2/\mathbb{Z}_2 \times S^1)/\mathbb{Z}_2$$ $A_1 \subset A_3$

$$\downarrow$$

$$(T^*S^2 \times S^1)/\mathbb{Z}_2$$ (Smooth)

# Warum Up

## 2 representative cases

**IIA** | **M**

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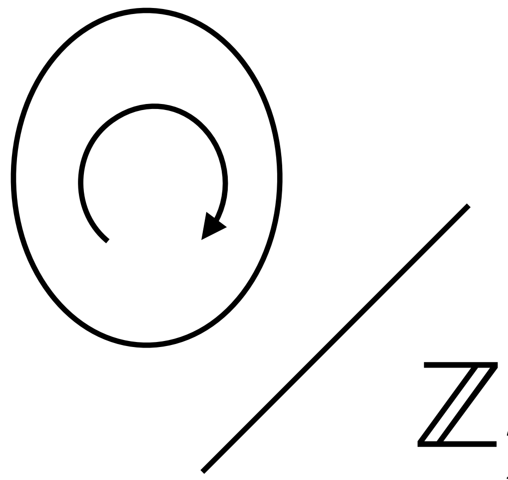
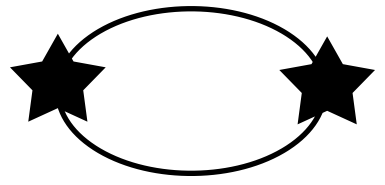
$S^2$



$\mathbb{Z}_2$

$=$

$A_1$ $A_1$



$S^2/\mathbb{Z}_2$

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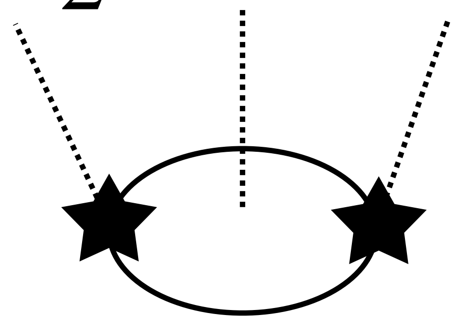

$A_1 \subset A_3 : \alpha_1^{A_3} + 2\alpha_2^{A_3} + \alpha_3^{A_3}$

# Warum Up

## 2 representative cases

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Orbi-Circle-Bundle

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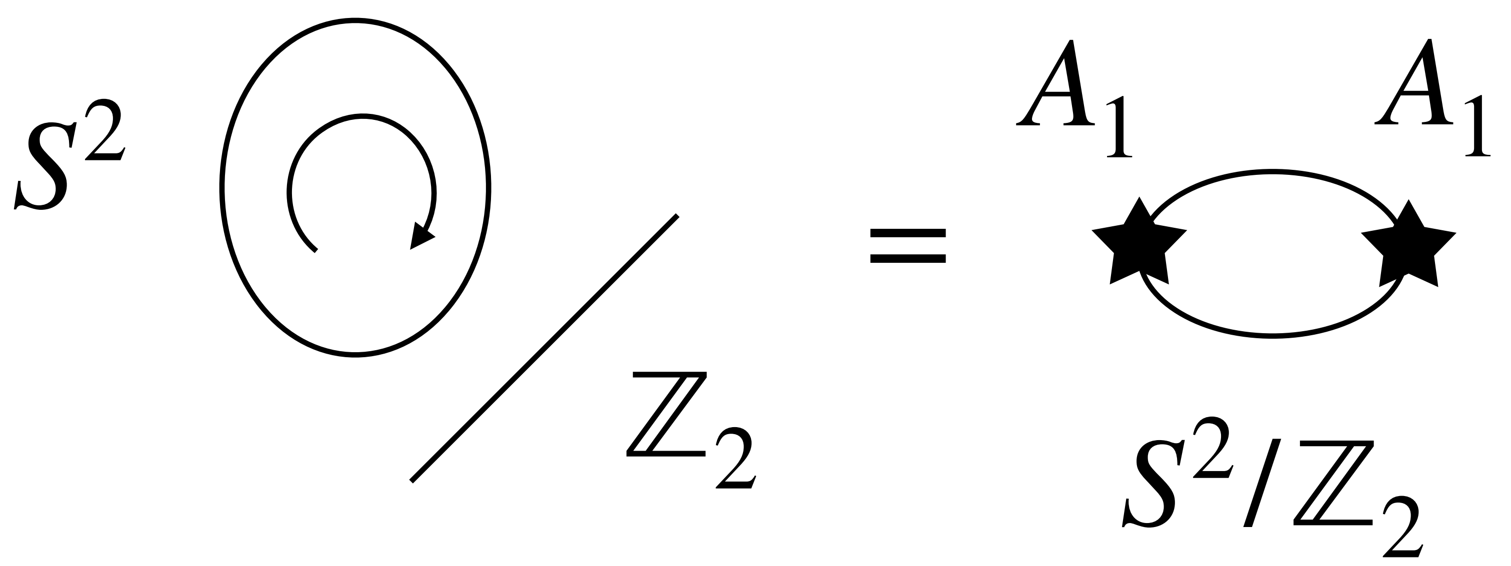


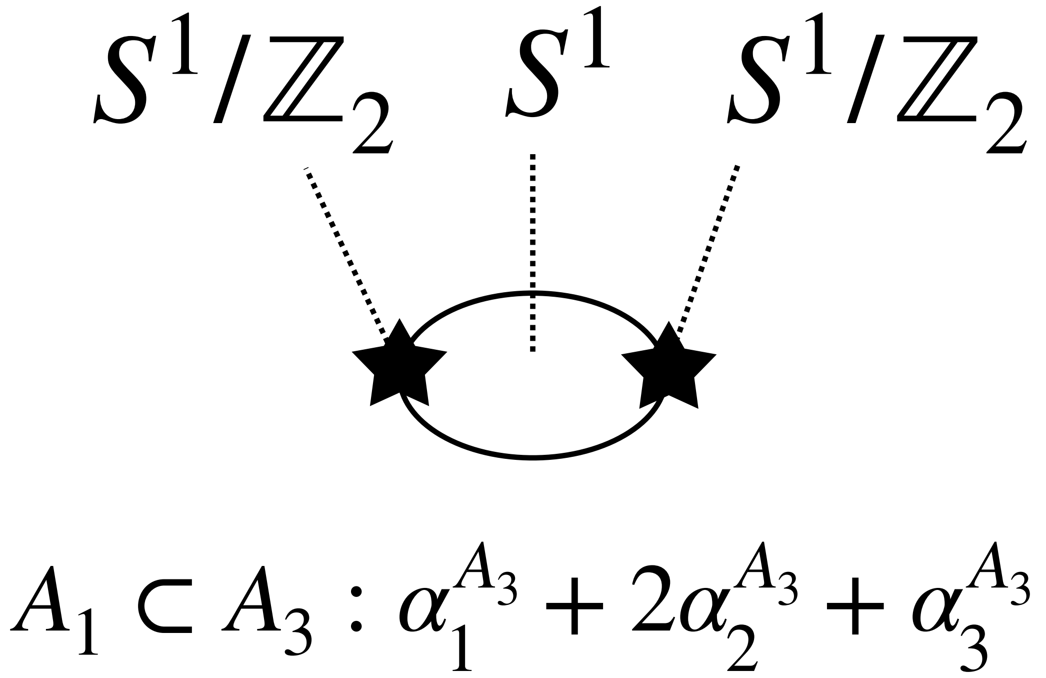

# Warum Up

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IIA: How do the D-branes on vanishing cycles couple to this RR-background?

E.g., D2-branes on vanishing 2-cycles.

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$$\mathcal{S}_{WZ}^{D4} \supset \int_{\mathbb{C}^2/\mathbb{Z}_N \times L} C_1^{RR} \wedge \mathrm{ch}_2(A^{D4}) = \frac{1}{2}\int_{\mathbb{C}^2/\mathbb{Z}_N \times L} C_1^{RR} \wedge F^{D4} \wedge F^{D4} \quad \Rightarrow \quad \text{1D CS-term:} \quad q\int_L A_1^{D4} \quad \text{with} \quad q \in \frac{1}{N}\mathbb{Z}$$

# Warum Up

## Formalization in K-theory

D-brane Charges take Values in K-theory: $K^0_\Gamma(\mathbb{C}^2) \cong K^0_\Gamma(\mathrm{pt}) \cong \mathrm{Rep}(\Gamma)$

(Here we have applied Kunneth to $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma$ and leave spacetime implicit)

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Reduced K-theory subgroup: $\bar{K}^0_\Gamma(\mathbb{C}^2) \cong \Lambda_{wt}(\mathfrak{g}) \supset \Lambda_{rt}(\mathfrak{g})$

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Euler Pairing: $\chi_{kl} = \int_{\mathbb{C}^2/\Gamma} \mathrm{ch}(V_k^{(cpct)} \otimes (V_l^{(cpct)})^\vee)\mathrm{Td}_{\mathbb{C}^2/\Gamma} = C_{kl}$

(cpct classes via Thom or sky scraper sheaf constructions, C Cartan matrix)

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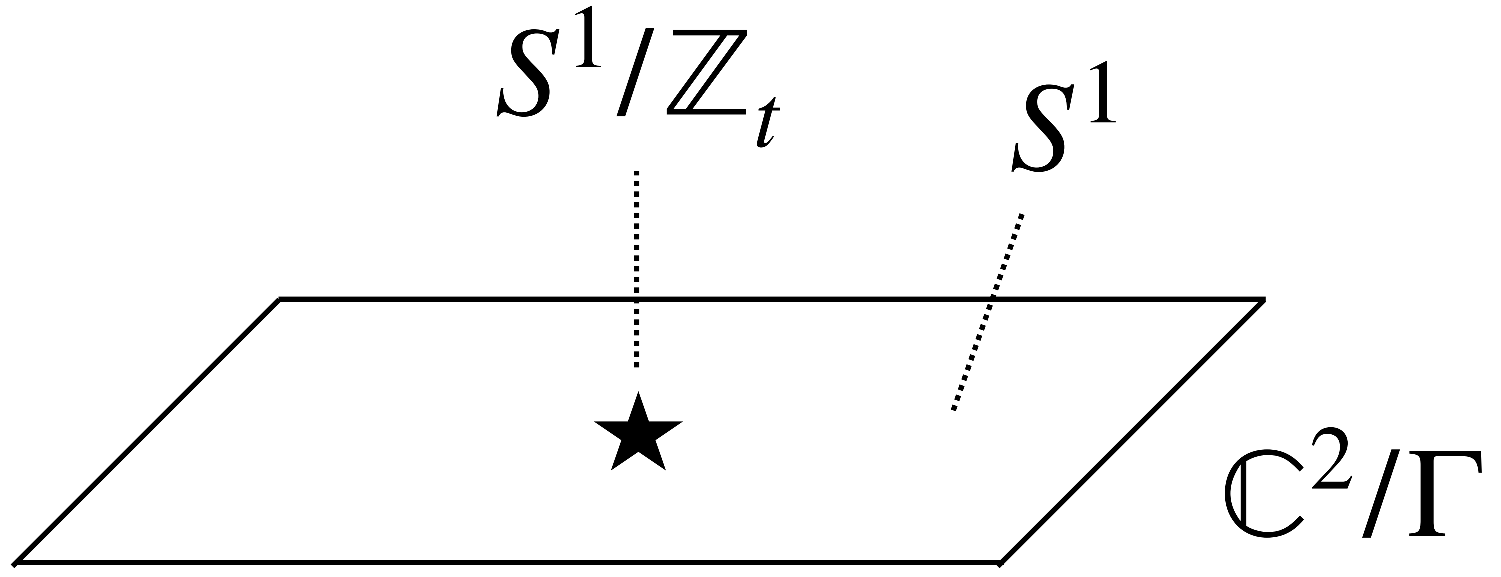
M-theory orbi-circle bundle:

$$g^2_{SYM} \sim g_s \ell_s \sim R_{11}\ell_s$$

$$\frac{t}{g^2_{SYM}} \sim \frac{1}{R_{11}/t}\frac{1}{\ell_s}$$



# Frozen ADE Singularities

## … in IIA

IIA Background: Geometry + Fractional 4-form RR-Flux

$$M^{10} = \mathbb{R}^{1,5} \times X^4_{n/d} \qquad X^4_{n/d} = ALE, ALF, ALG, ALH \quad \text{with} \quad \int_\partial C_3^{RR} = \frac{n}{d} \mod 1$$

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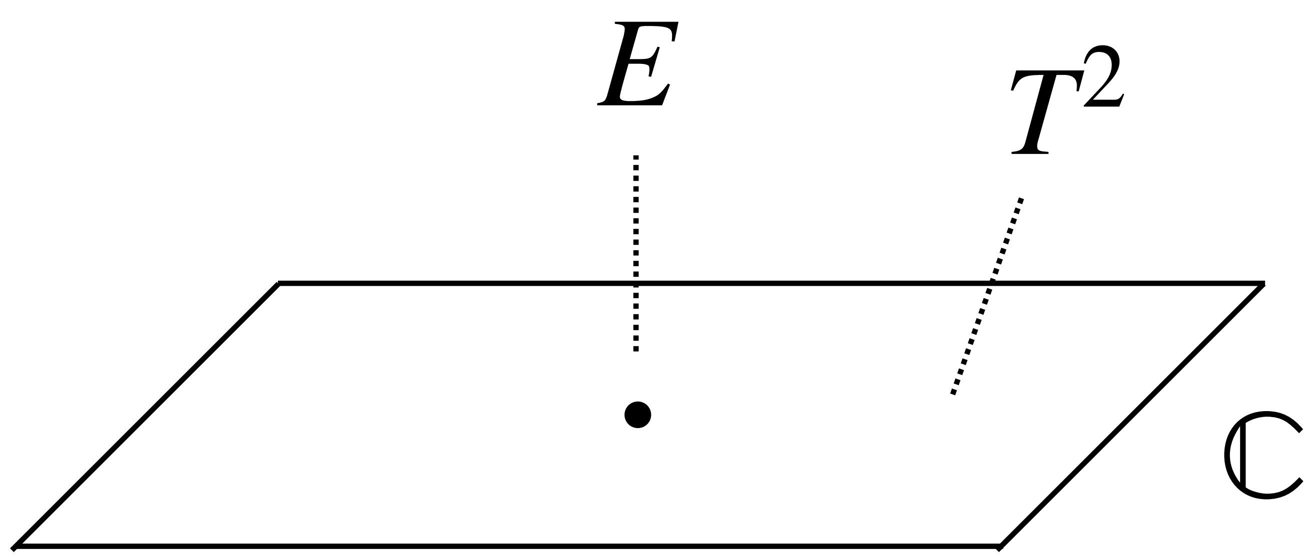
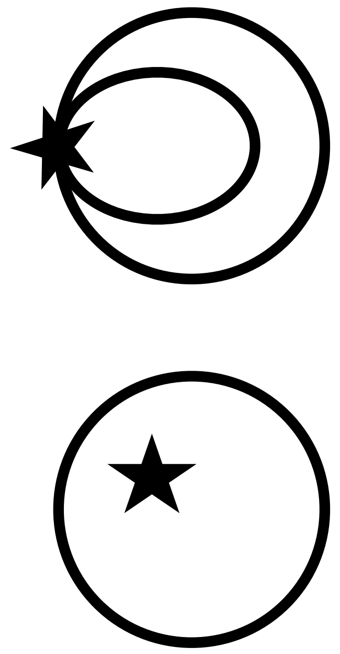



$E$ of $I_n$-type

all other cases



★ (ADE Singularity)

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K-theory : D2 $\subset$ D4-$\overline{\text{D4}}$

$$\int_{\mathbb{C}^2/\Gamma_{\mathrm{ADE}} \times L_1} C_3^{\mathrm{RR}} \wedge \mathrm{Tr}(F_2 - \overline{F}_2) = \frac{n}{d} \int_{L_1} \mathrm{Tr}(A_1 - \overline{A}_1) = \frac{n}{d} d_k \int_{L_1} (a_1 - \overline{a}_1)\,,$$

Quantization condition on 1D CS-term: $\quad \dfrac{n}{d} d_k = 0 \quad \mathrm{mod}\, 1\,.$

# Frozen ADE Singularities

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When $(n/d)d_k \neq 0$ mod 1 then the particle can remain only in the spectrum as a particle relative to a string:

$$\frac{n}{d} d_k \int_{L_1} (a_1 - \bar{a}_1) + \frac{n}{d} d_k \int_{\Sigma_2} B_2 \,,$$

To determine whether this is indeed the case we need to check for the presence of **confining strings**!

Interestingly we will find **confining strings**, and these will lie beyond K-theory.

# Frozen ADE Singularities

## Confining Strings

IIA Background: Geometry + Fractional 4-form RR-Flux

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NS5-branes carry the topological coupling: $\int H_3^{NS} \wedge C_3^{RR}$

$\Rightarrow$ NS5-anti-NS5-stacks on $\Sigma_2 \times \mathbb{C}^2/\Gamma$ result in objects carrying fractional fundamental string charge:

$$\frac{n}{d} \int_{\Sigma_2} B_2^{NS}$$

Fractional fundamental strings? We need to know the tension.

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## Confining Strings

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$\boxed{\text{Dual Magnetic Objects? D6-anti-D6 configurations?}}$

Fractional fundamental strings? We need to know the tension.

# Frozen ADE Singularities

## Domain walls and Chern-Simons Invariants

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Consider D8-anti-D8 system on $M_5 \times \mathbb{C}^2/\Gamma$:

$$\int_{\mathbb{C}^2/\Gamma_{\mathrm{ADE}} \times M_5} C_5^{\mathrm{RR}} \wedge \left(\mathrm{ch}_2(F_2) - \mathrm{ch}_2(\overline{F}_2)\right) = q \int_{M_5} C_5^{\mathrm{RR}} , \tag{2.20}$$

where $q$ is a Chern-Simons invariant

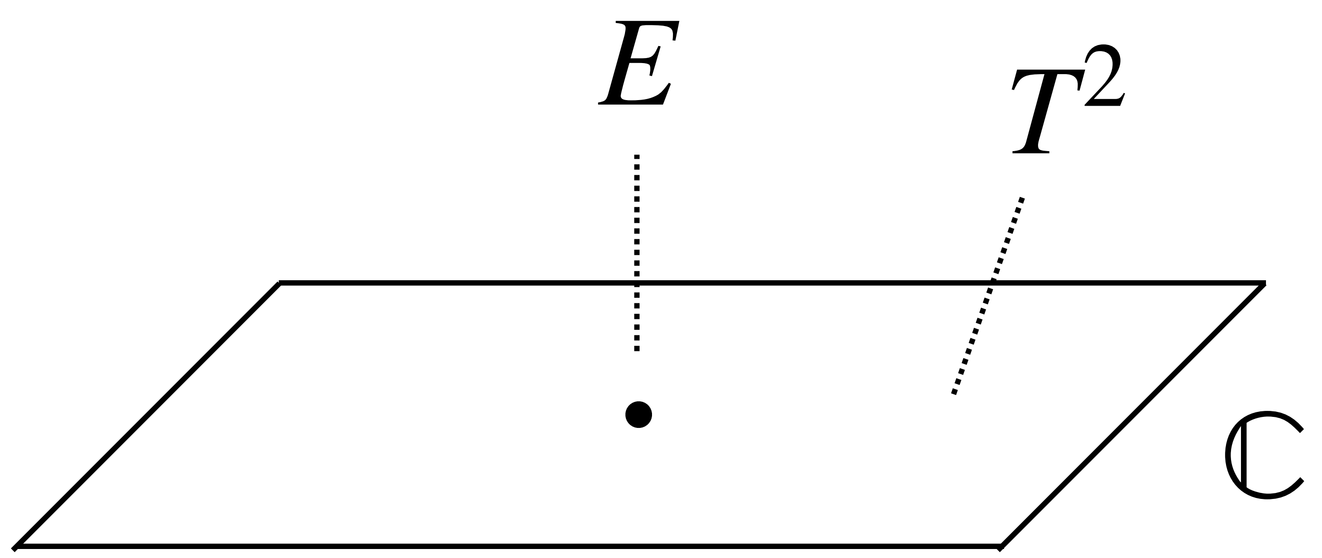
$$q = \int_{S^3/\Gamma_{\mathrm{ADE}}} \left(\mathrm{CS}(A_1) - \mathrm{CS}(\overline{A}_1)\right) , \tag{2.21}$$

of the asymptotic boundary $\partial X = S^3/\Gamma_{\mathrm{ADE}}$. These fractional D4-branes are domain walls between frozen phases and it is well-known that $n/d$ which is turned on by passing across such a wall is indeed a Chern-Simons invariant.

# Frozen ADE Singularities

## Which particles are confined? Who are the confining strings?

$X_4 = ALG :$




with $\displaystyle \int_{\partial X_4} C_3^{RR} = \frac{n}{d} \mod 1$

Elliptic Fibration with 4-form RR-flux

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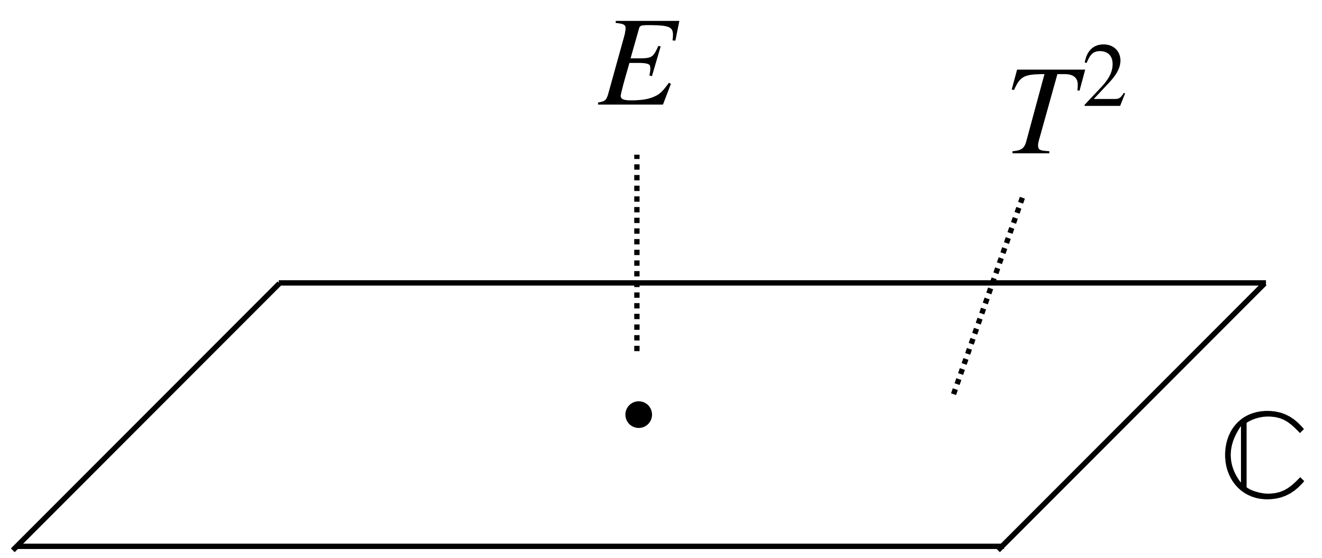



with $\displaystyle\int_{\partial X_4} C_3^{RR} = \frac{n}{d} \mod 1$

Elliptic Fibration with 4-form RR-flux

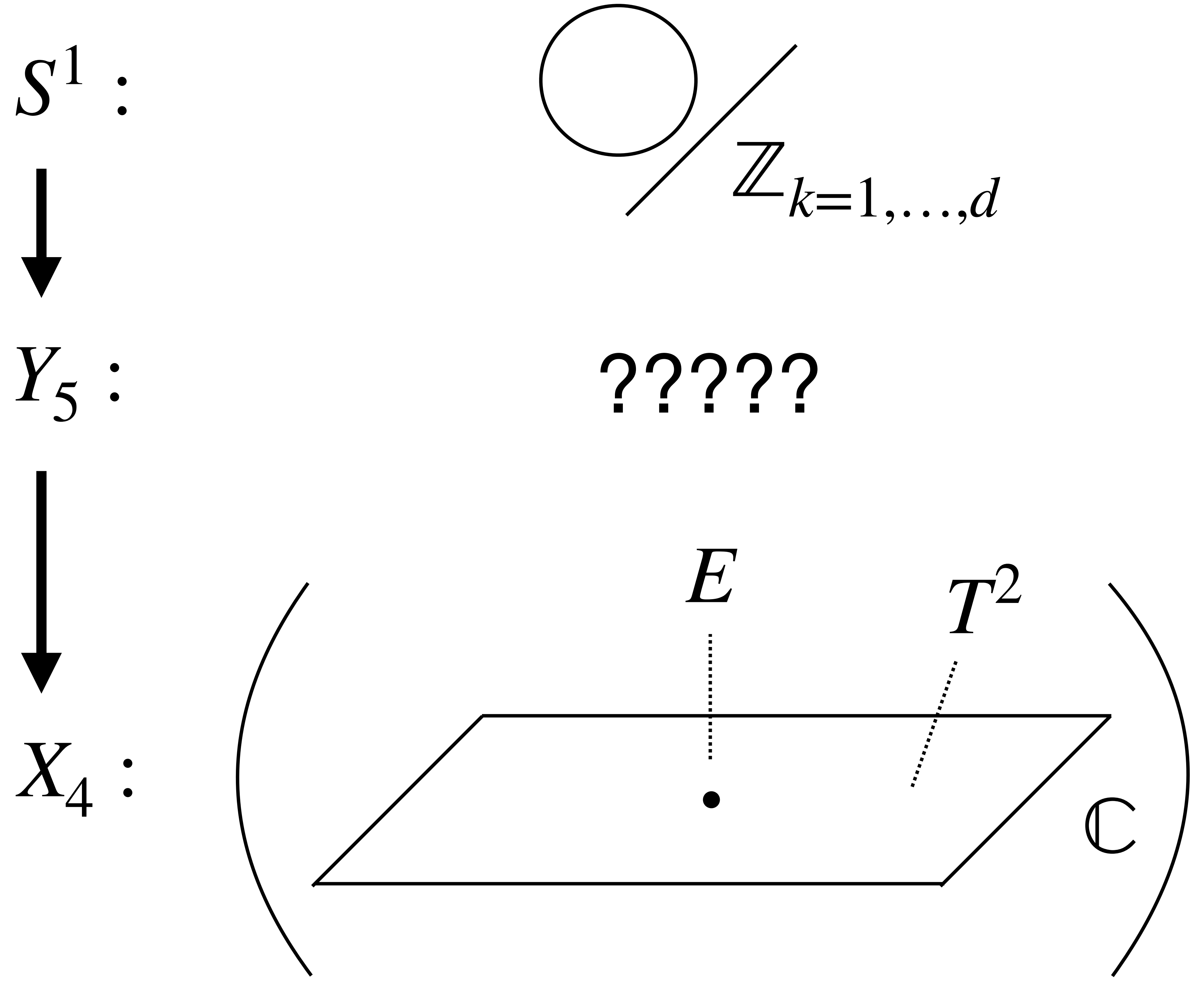
$\Downarrow$ Double T-duality

Elliptic Fibration with 2-form RR-flux: with $\displaystyle\int_{S^1 \subset \partial X_4} C_1^{RR} = \frac{n}{d} \mod 1$

Q: What is the M-theory Circle Orbi-Bundle?

# Frozen ADE Singularities

## Which particles are confined? Who are the confining strings?

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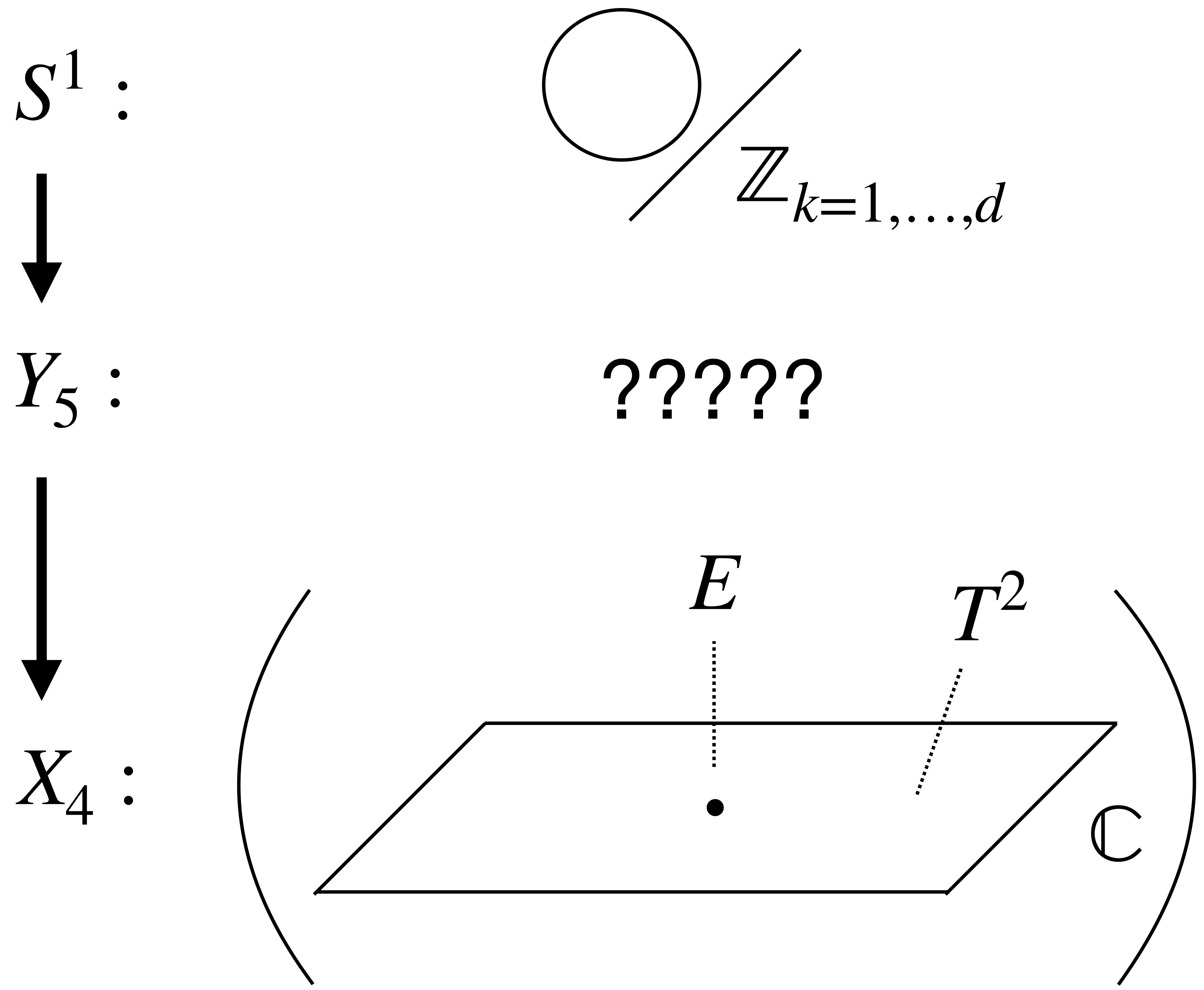


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$$T^2 = \sum_i d_i \alpha_i$$

$d_i \dfrac{n}{d} = 0 \quad \text{mod } 1 \quad \Rightarrow$ $\alpha_i$ can be blown up

$d_i \dfrac{n}{d} \neq 0 \quad \text{mod } 1 \quad \Rightarrow$ $\alpha_i$ can not be blown up + exceptional fiber

# Frozen ADE Singularities

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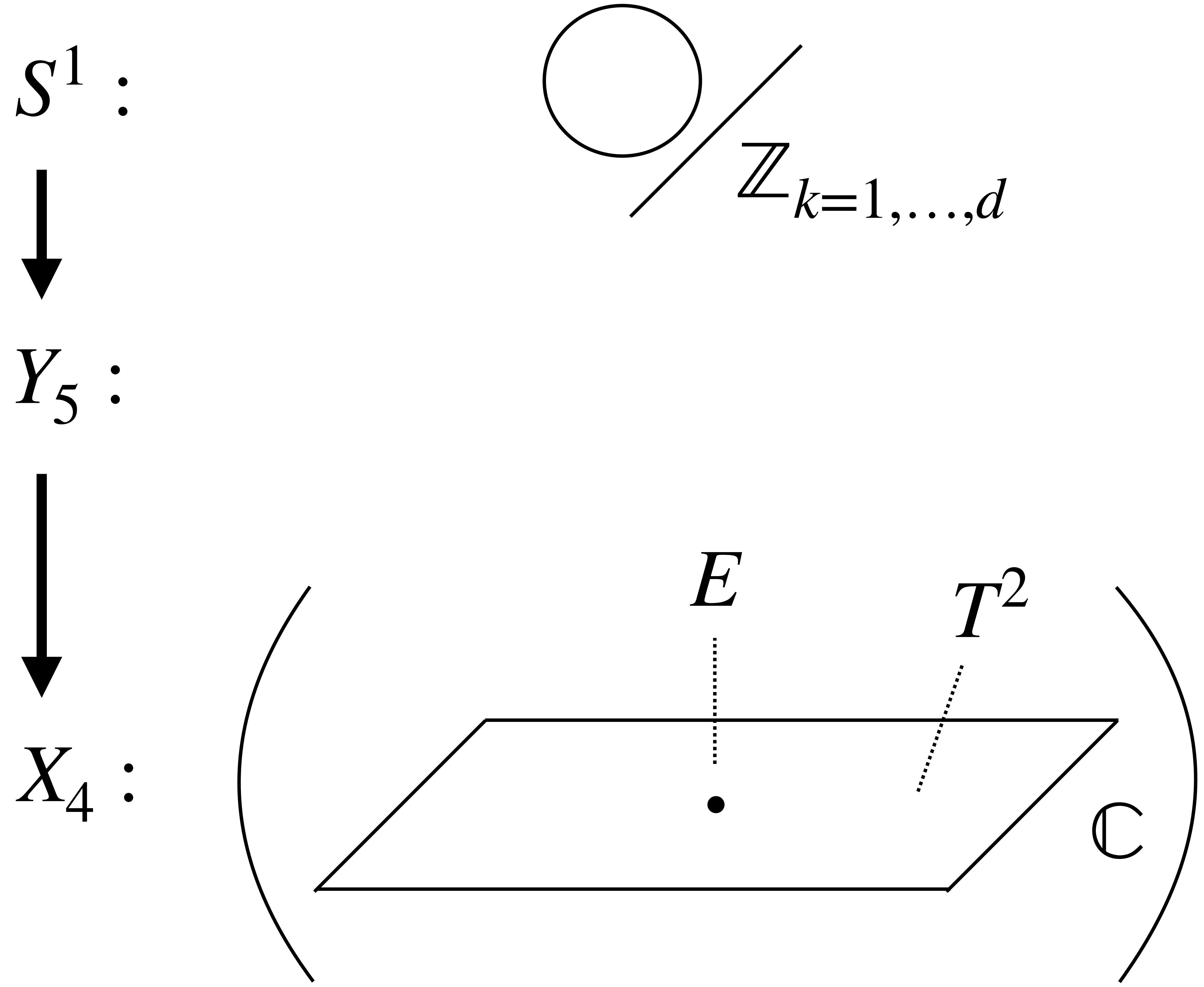


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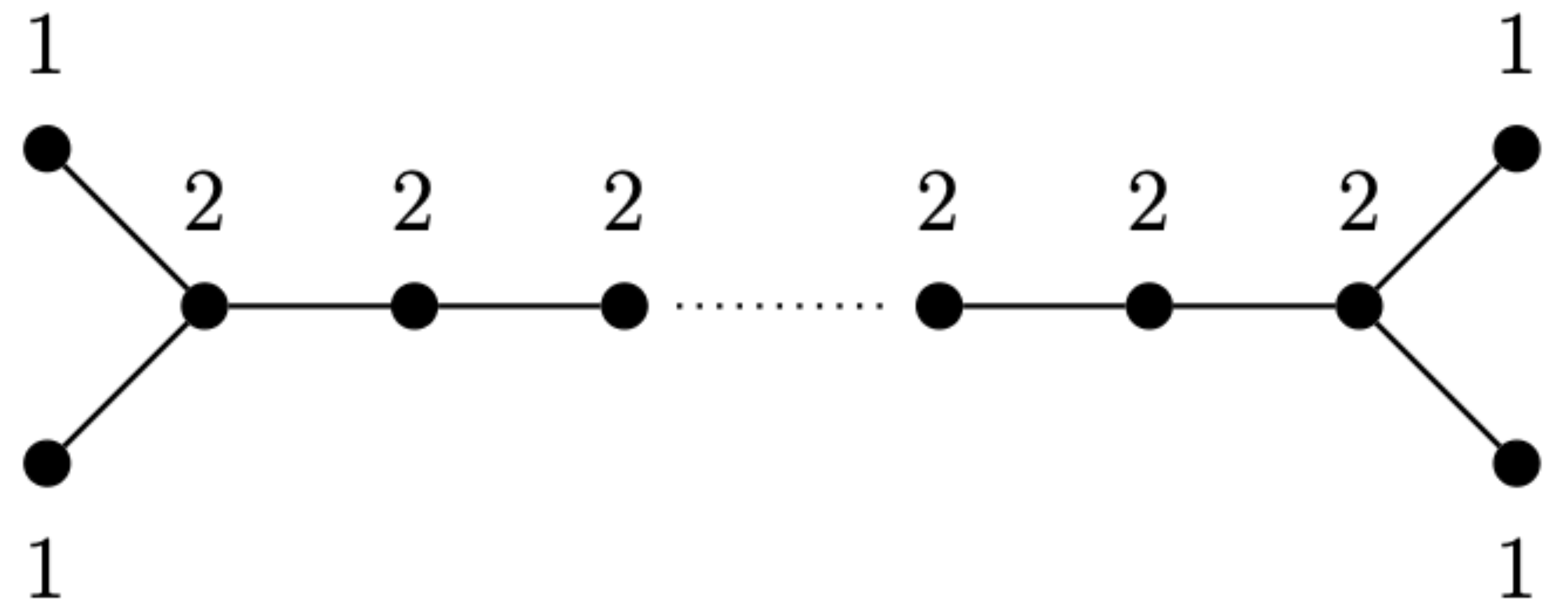
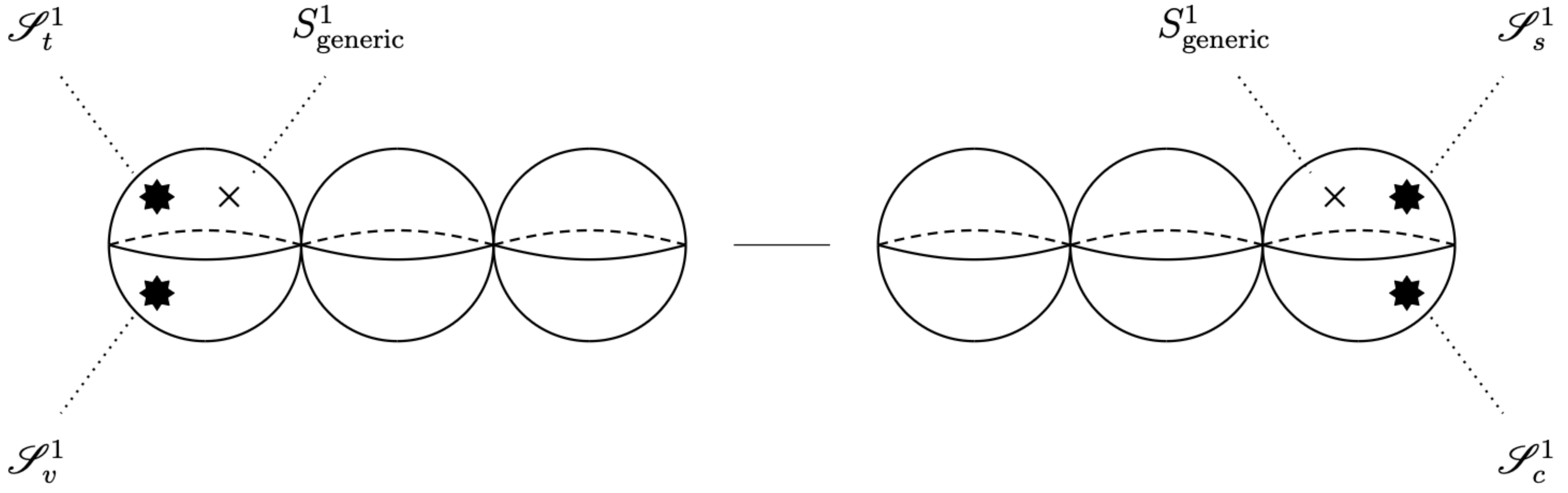
# Frozen ADE Singularities

## Which particles are confined? Who are the confining strings?

Example: $I_n^*$ with $\int C_3^{RR} = \int C_1^{RR} = \frac{1}{2}$



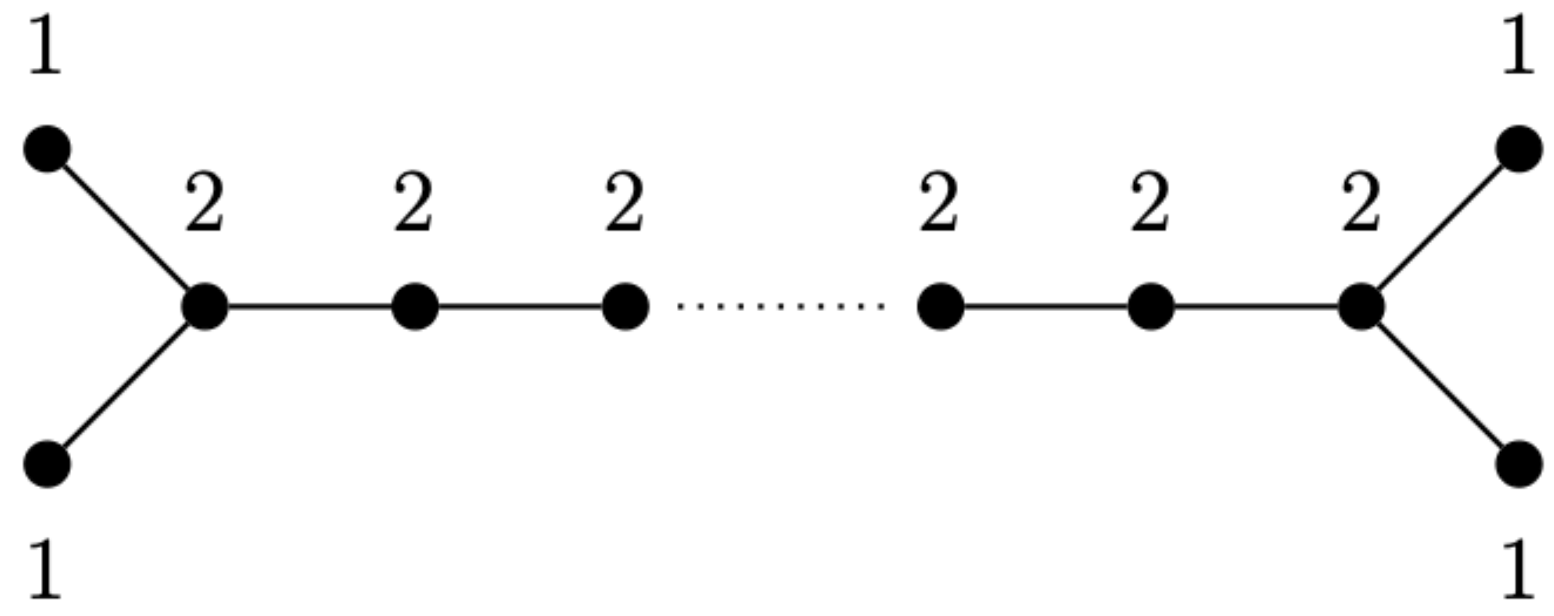

$(\mathbb{C}^2 \times S^1)/\mathbb{Z}_2$

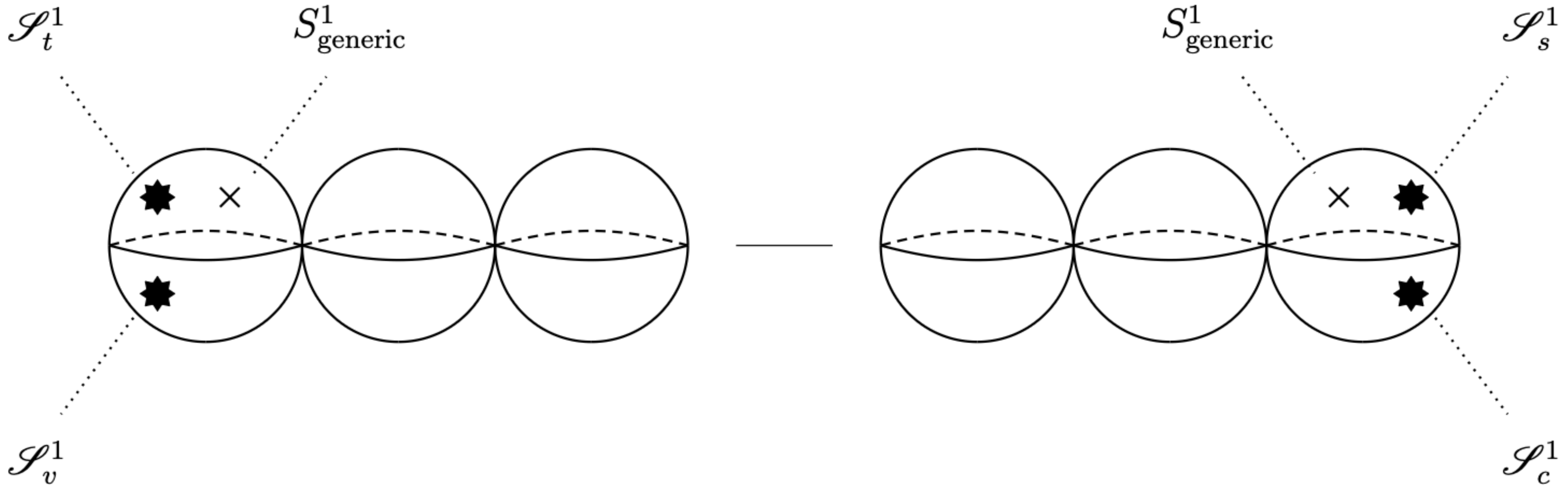
$S^1/\mathbb{Z}_2 = \mathcal{S}$

# Frozen ADE Singularities

## Which particles are confined? Who are the confining strings?

Example: $I_n^*$ with $\int C_3^{RR} = \int C_1^{RR} = \frac{1}{2}$




$$(\mathbb{C}^2 \times S^1)/\mathbb{Z}_2$$

$$S^1/\mathbb{Z}_2 = \mathcal{S}$$

M2 on $S^1/\mathbb{Z}_2$

⇔ Fractional Fundamental String

⇔ Confining String

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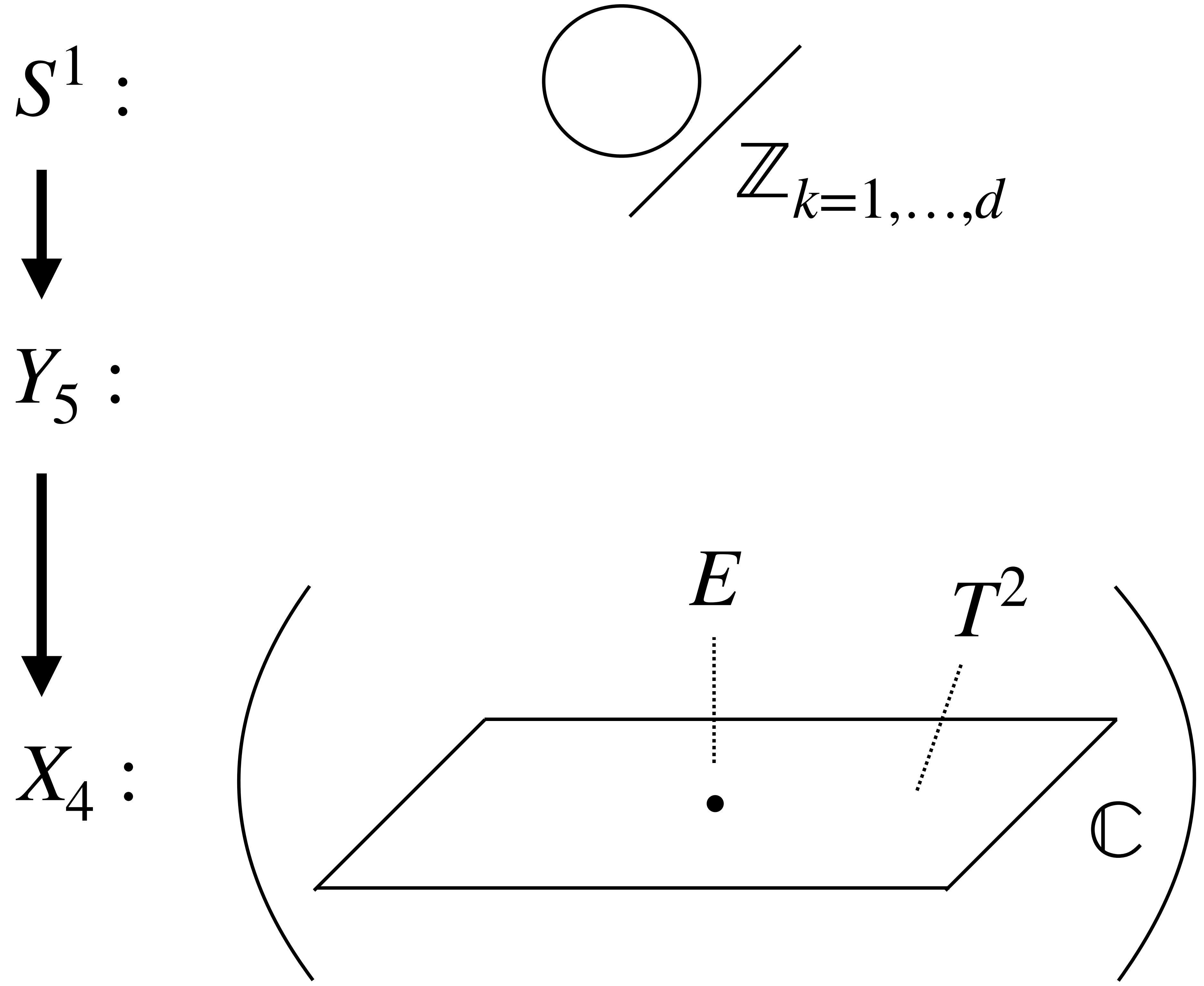


with $\displaystyle\int_{S^1\subset\partial X_4} C_1^{RR} = \frac{n}{d}$ mod 1

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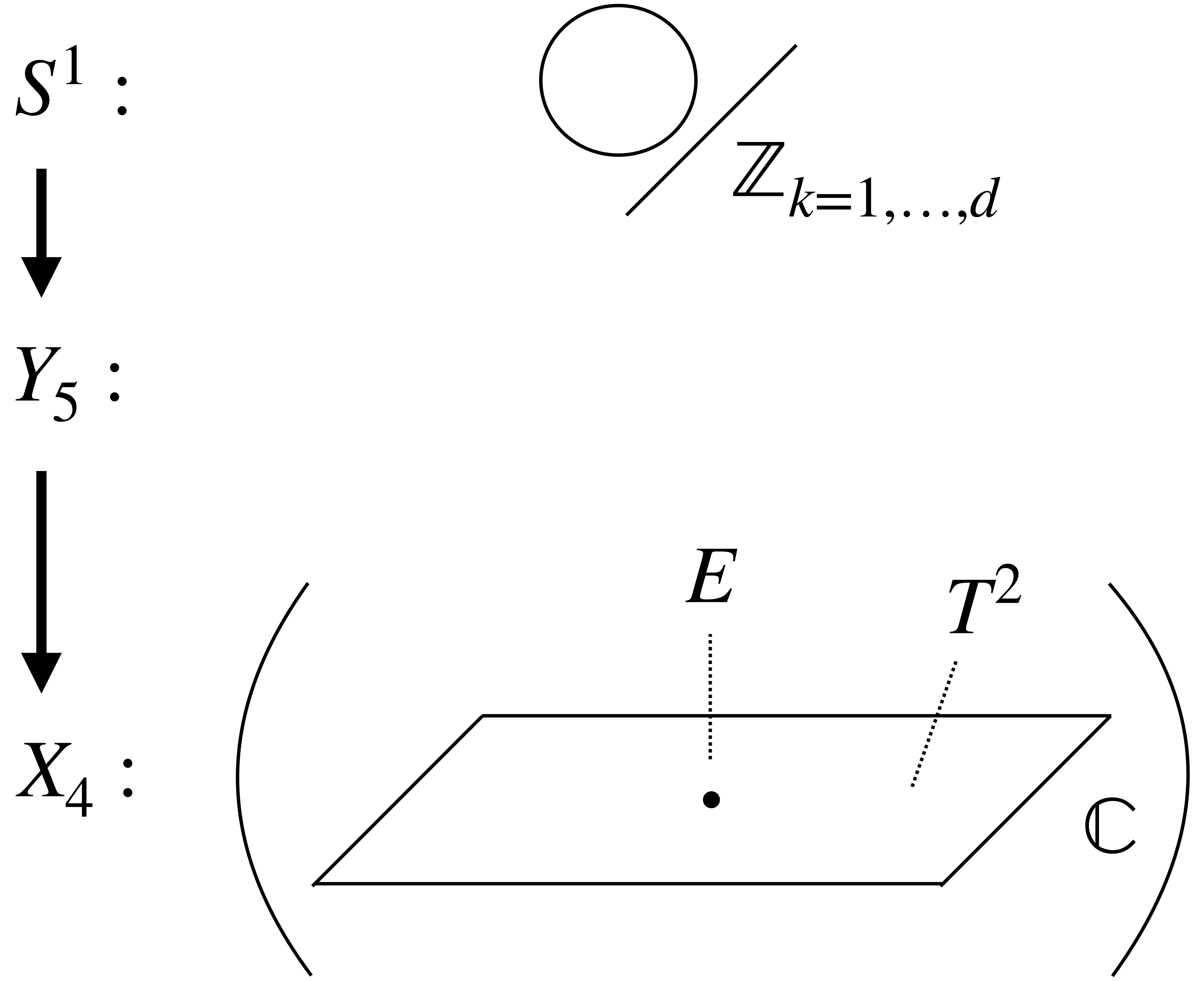



with $\int_{S^1 \subset \partial X_4} C_1^{RR} = \frac{n}{d}$ mod 1

M2, M5 on $S^1/\mathbb{Z}_2$

$\Leftrightarrow$ Fractional Strings / D-branes

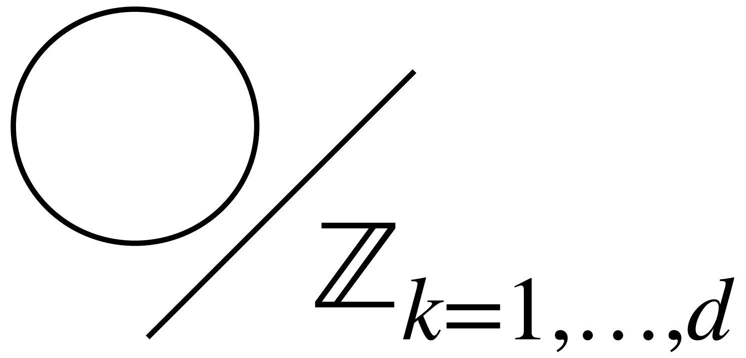
Q: How to characterize the unconfined particles?

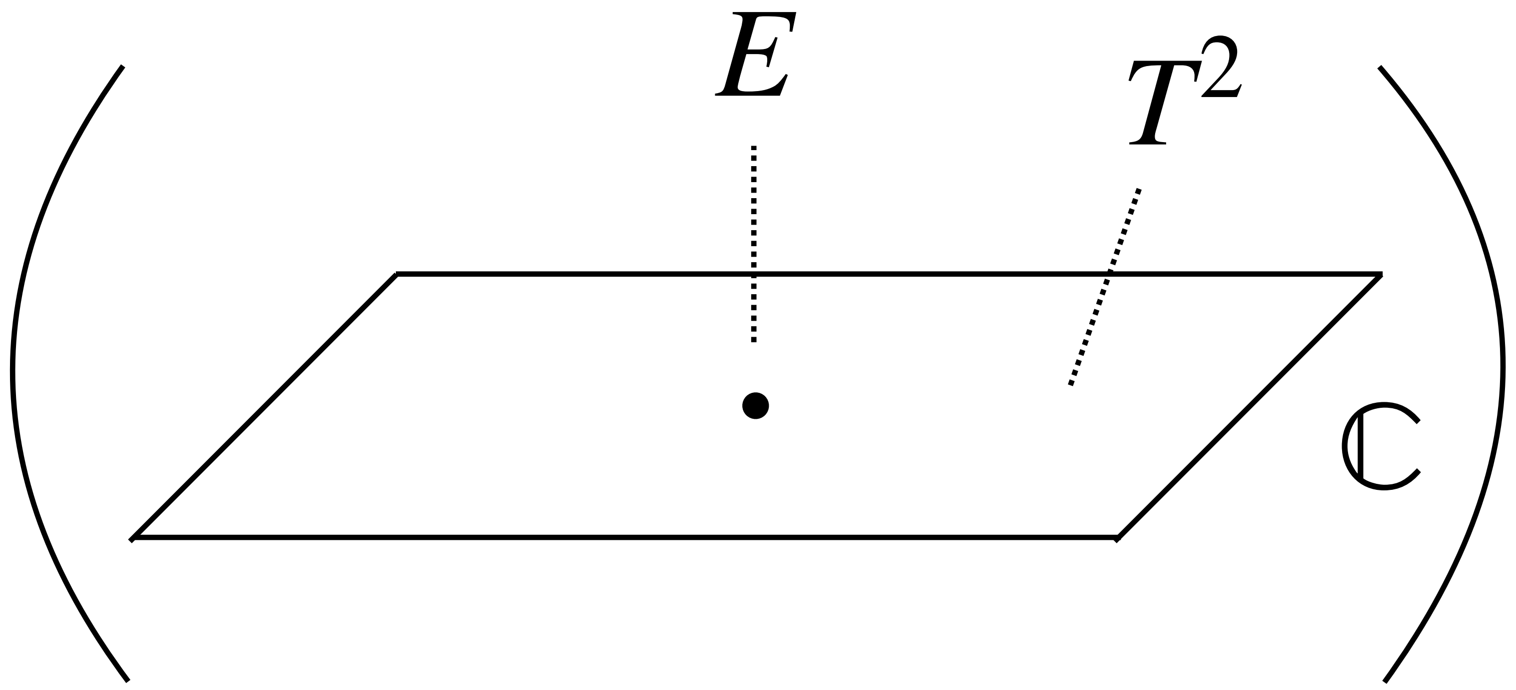
(With respect to the original K-theory charge lattice)

# Frozen ADE Singularities

## Which particles are confined? Who are the confining strings?

$$
\begin{array}{l}
S^1 : \\
\downarrow \\
Y_5 : \\
\downarrow \\
X_4 :
\end{array}
$$




$$Z \xrightarrow{p_{n/d}} \widetilde{X}_{n/d} \xrightarrow{q_{n/d}} X$$

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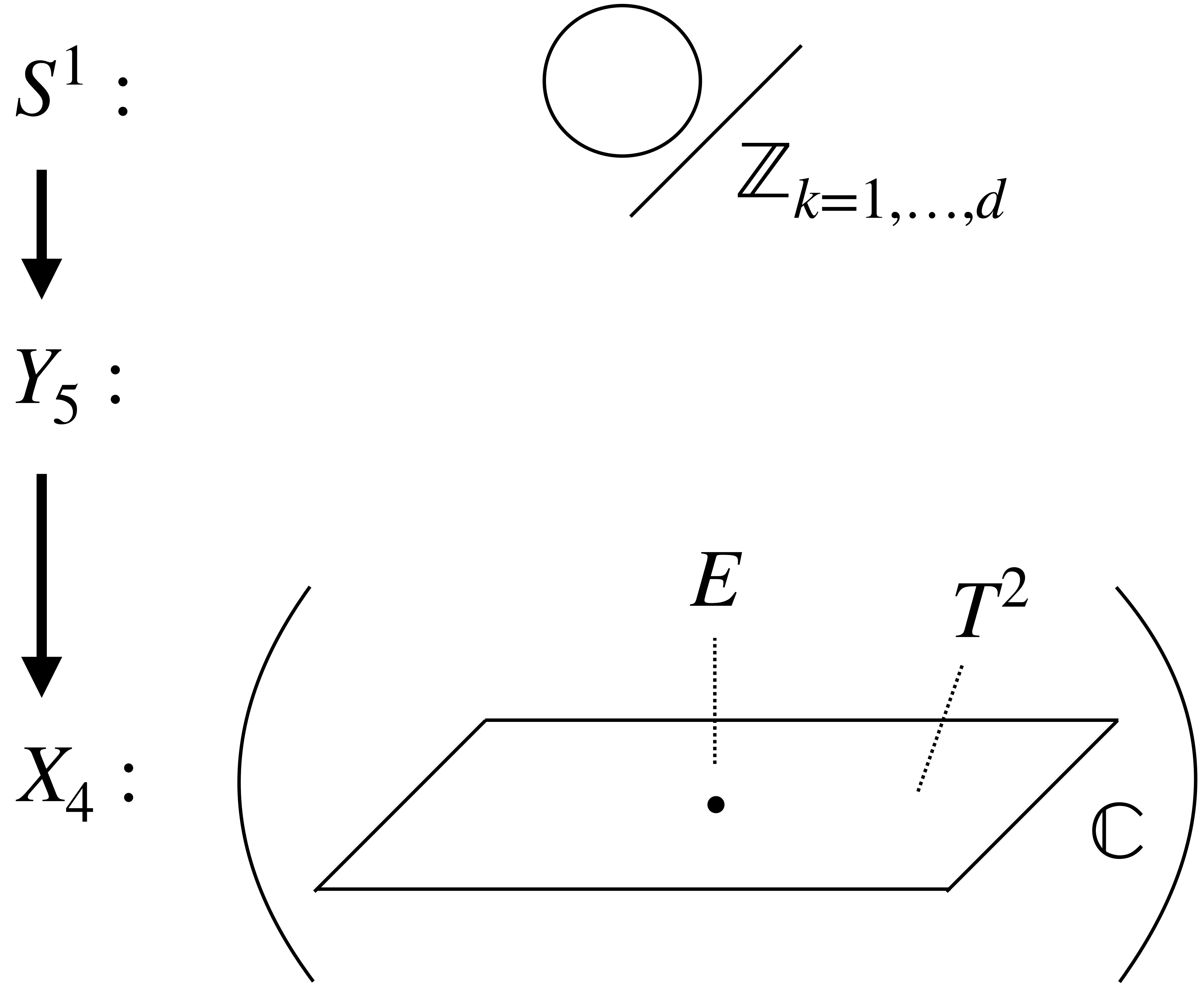


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$$\bar{K}^0(Z) = \Big\{ \sum_i n_i W_i \in K^0(Z) \,|\, \sum_i n_i \,\mathrm{rk}\, W_i = 0 \Big\} \cong \Lambda_{\mathrm{wt}}(\mathfrak{g}_{\mathrm{ADE}})$$

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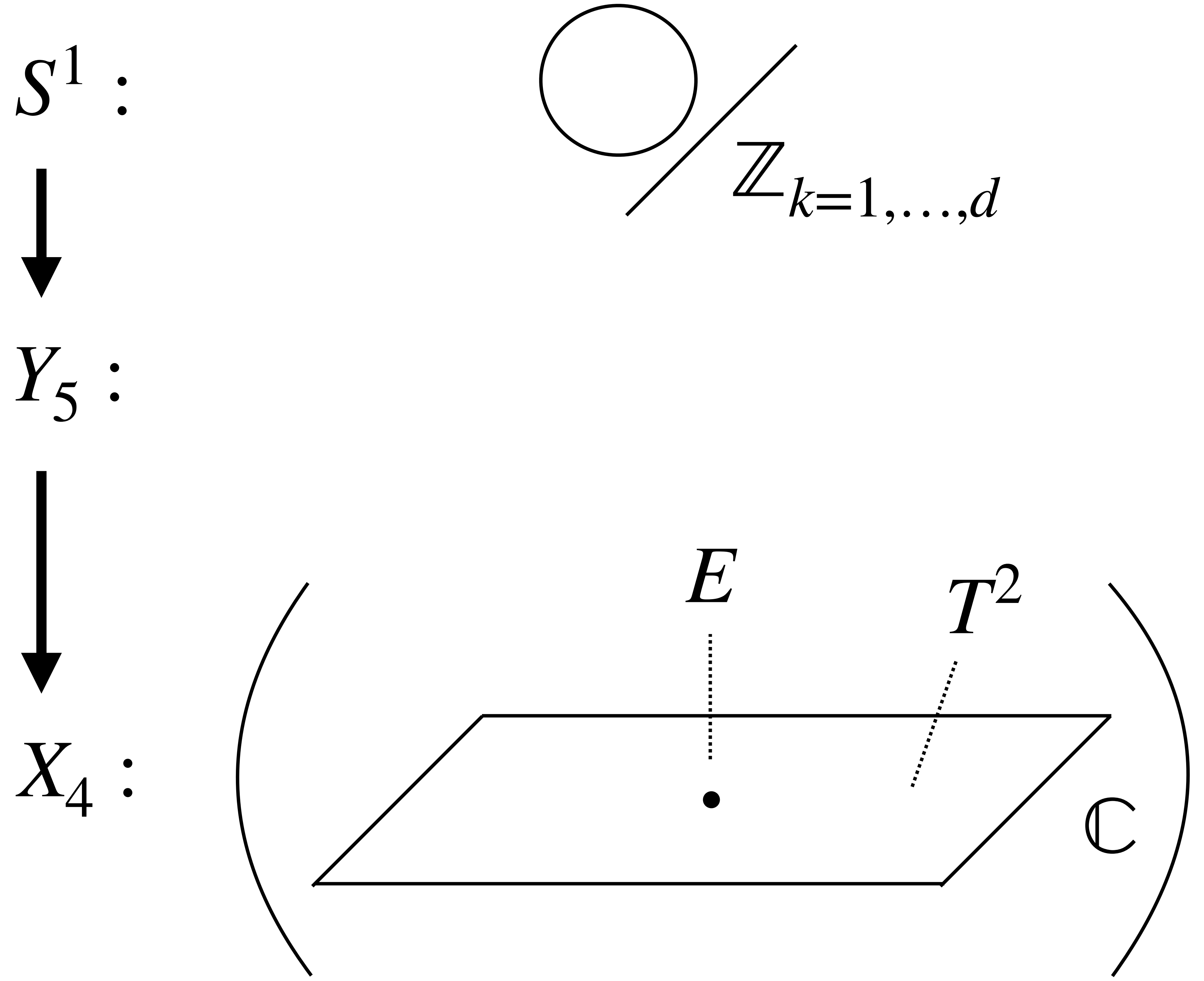


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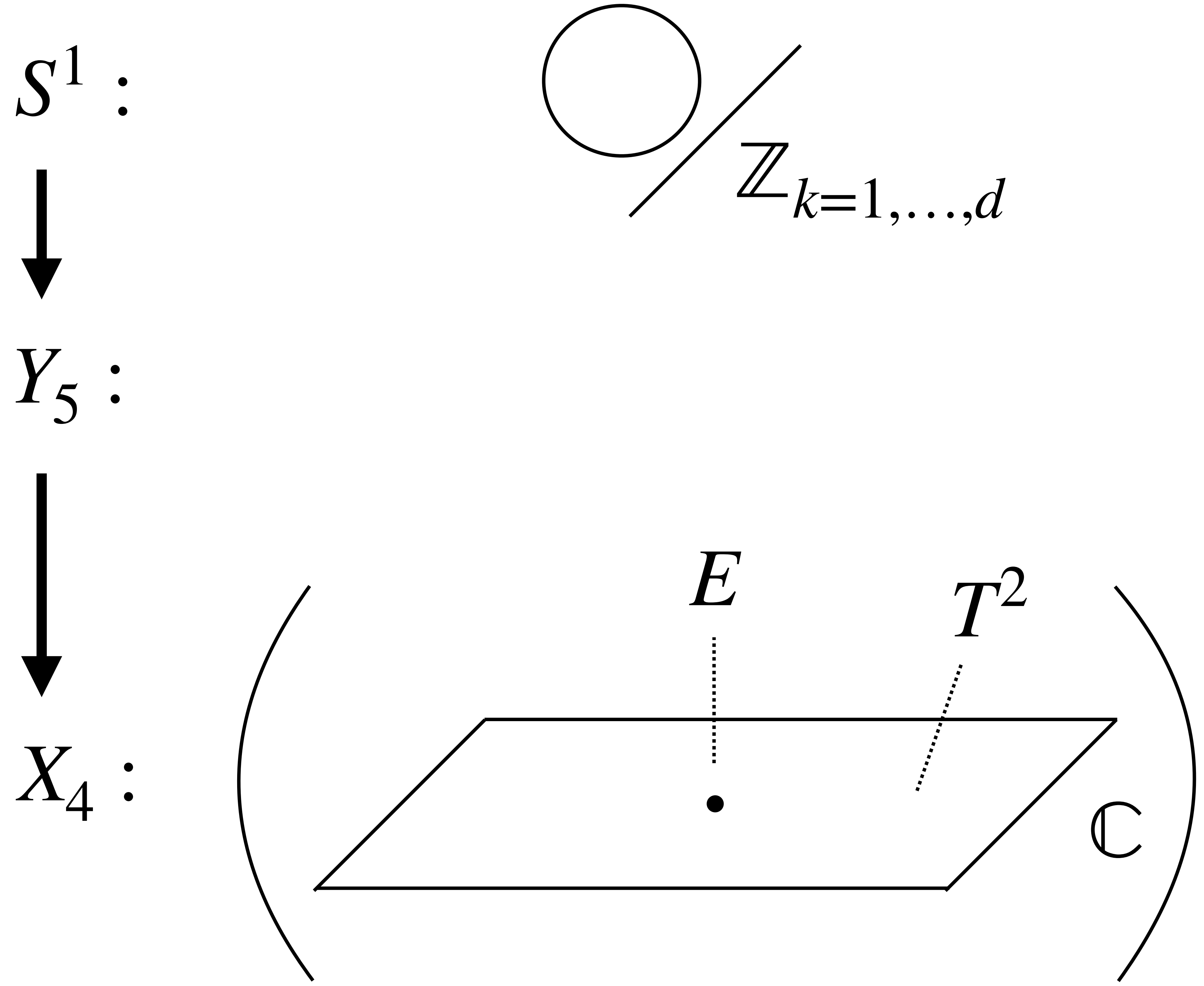


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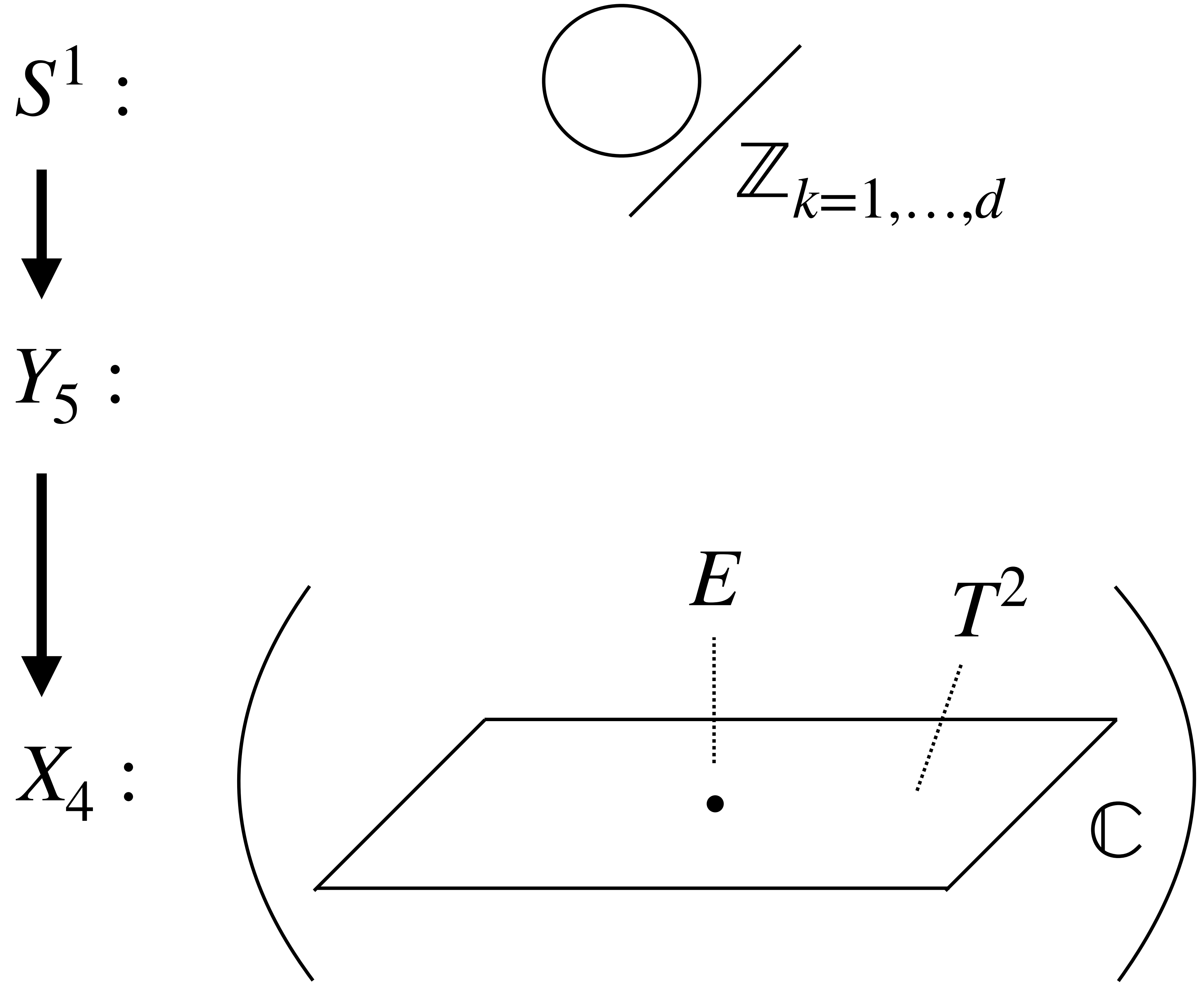


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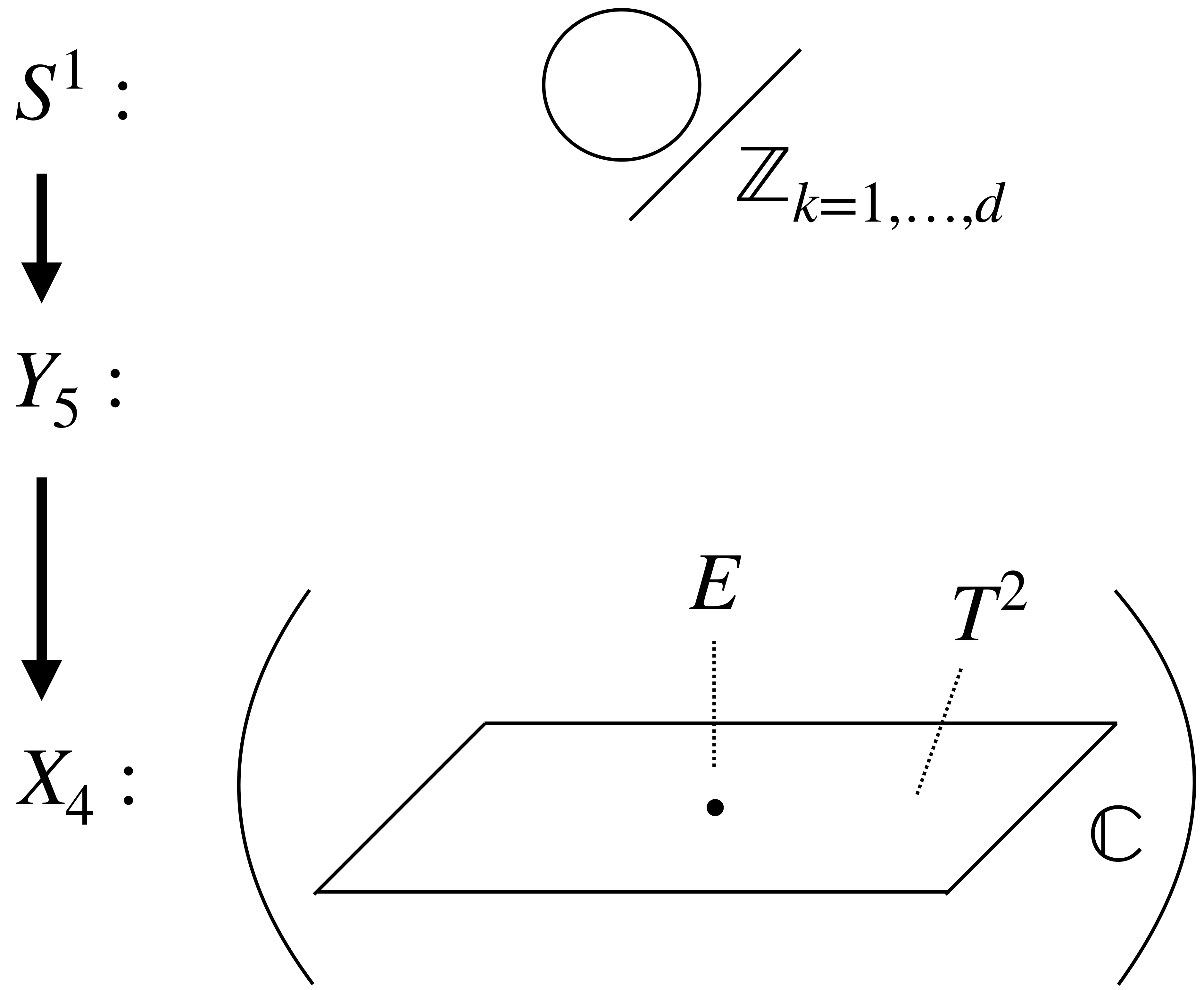


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$$\bar{K}^0(\widetilde{X}_{n/d}) \cong \operatorname{Im} p^*_{n/d} = \{ V \in \bar{K}^0_{\Gamma_{\mathrm{ADE}}}(\mathbb{C}^2) \,|\, \forall\, W \in \mathcal{B}_{n/d} :\; \chi(V,W) = 0 \} \subset \bar{K}^0_{\Gamma_{\mathrm{ADE}}}(\mathbb{C}^2)$$

# Recap

## … a lot has happened

Elliptic Local Model

Step 1: IIA Background

$$M^{10} = \mathbb{R}^{1,5} \times X^4_{n/d} \qquad X^4_{n/d} = ALG \quad \text{with} \quad \int_\partial C_3^{RR} = \frac{n}{d} \mod 1$$

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D4-$\overline{\text{D4}}$-System: Electric Particles + 1D CS-terms

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Step 2: Explicit Construction as fractional 1/2 BPS objects via double T-duality

Step 3: Uplift to M-theory? ⇒ Frozen ADE singularities in M-theory

# M-theory Uplifts

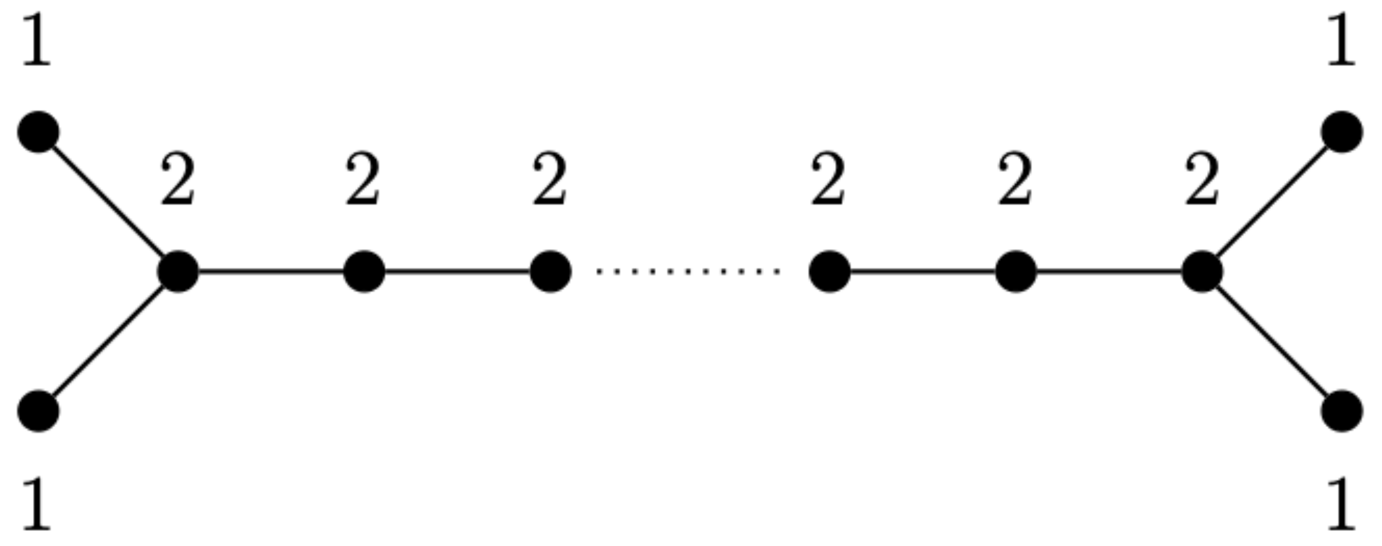
## Fractional D4s vs. Fractional M5s

IIA Background

$$M^{10} = \mathbb{R}^{1,5} \times X^4_{n/d} \qquad X^4_{n/d} = ALG \quad \text{with} \quad \int_\partial C_3^{RR} = \frac{n}{d} \mod 1$$

Fractional D4s are constrained by Rep$(\Gamma)$

Not all fractional D4s lift to fractional M5s



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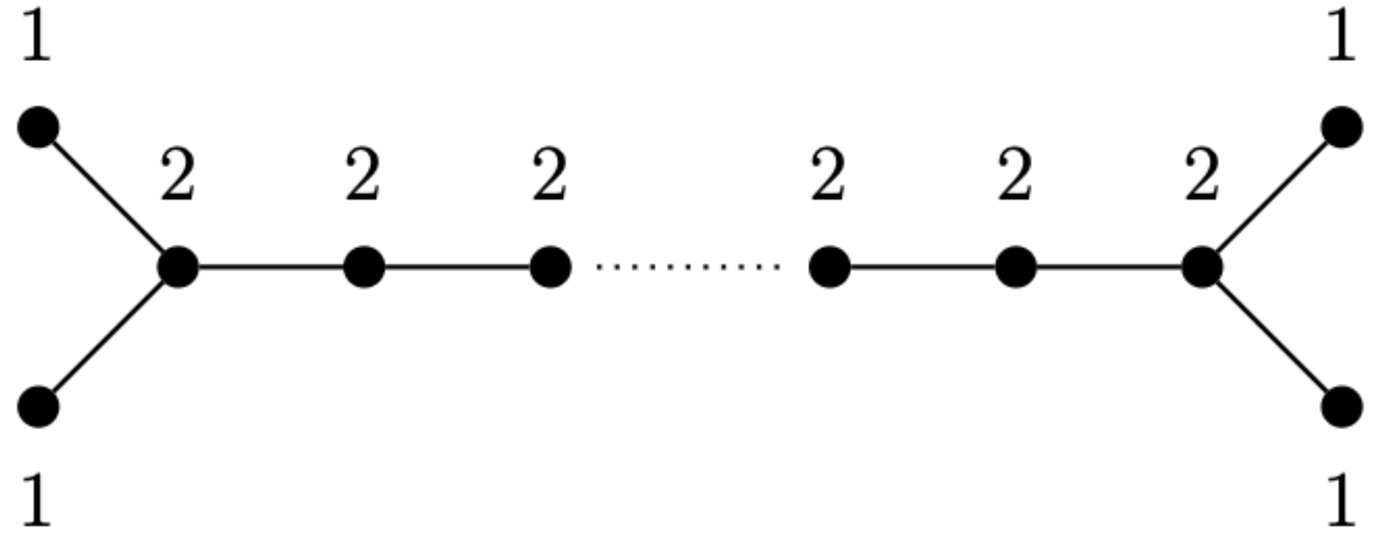


Full D4: Regular representation $R_{reg} = \bigoplus_i \mathrm{dim}(R_i) R_i$

Fractionation: Which splits lift? $R_{reg} = R_L \oplus R_R$

# M-theory Uplifts

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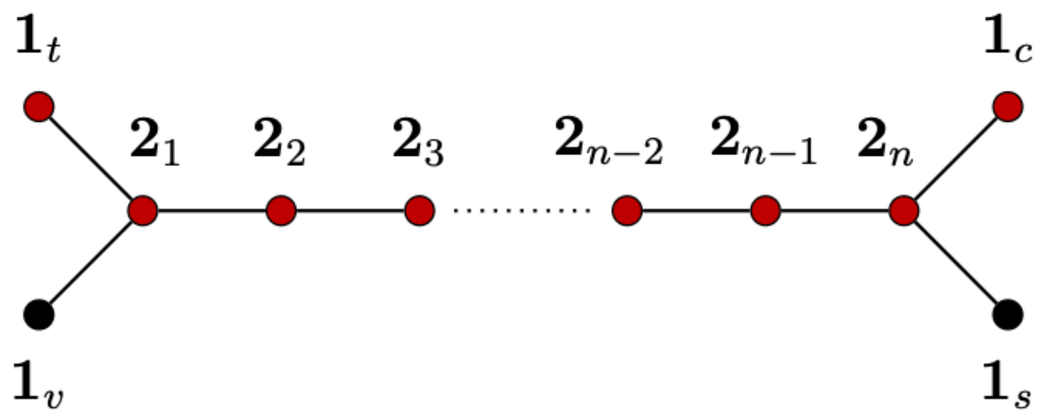
$$\int_{S^3/\Gamma_{\mathrm{ADE}}} C_3^{\mathrm{RR}} = 0 \;\Big|\; \int_{S^3/\Gamma_{\mathrm{ADE}}} C_3^{\mathrm{RR}} = \frac{\mathrm{rk}(R_L)}{|\Gamma_{\mathrm{ADE}}|} \;\Big|\; \int_{S^3/\Gamma_{\mathrm{ADE}}} C_3^{\mathrm{RR}} = 0$$

$$dH_7 = F_2 \wedge F_6$$

$$B^\perp_{n/d} \subset \bar{K}^0_\Gamma(\mathbb{C}^2)$$

$$\Lambda^{(\mathrm{M5})}_{n/d} \subset (\mathcal{B}^\perp_{n/d})^\perp$$

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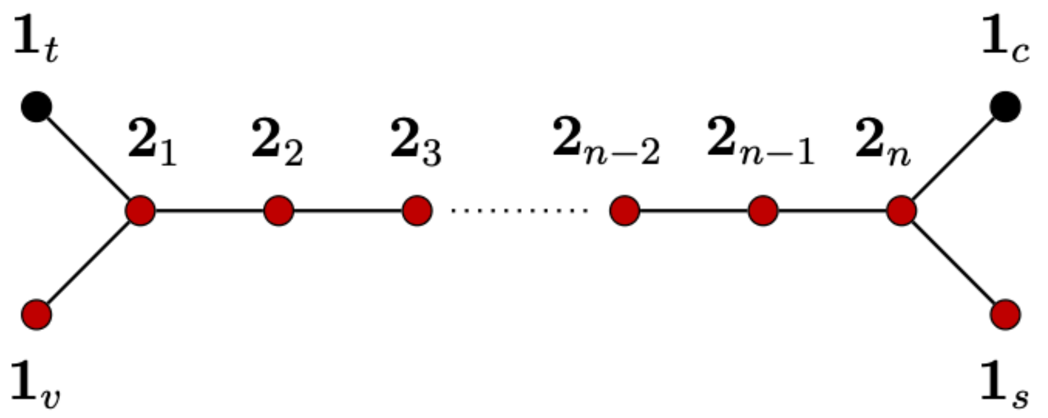


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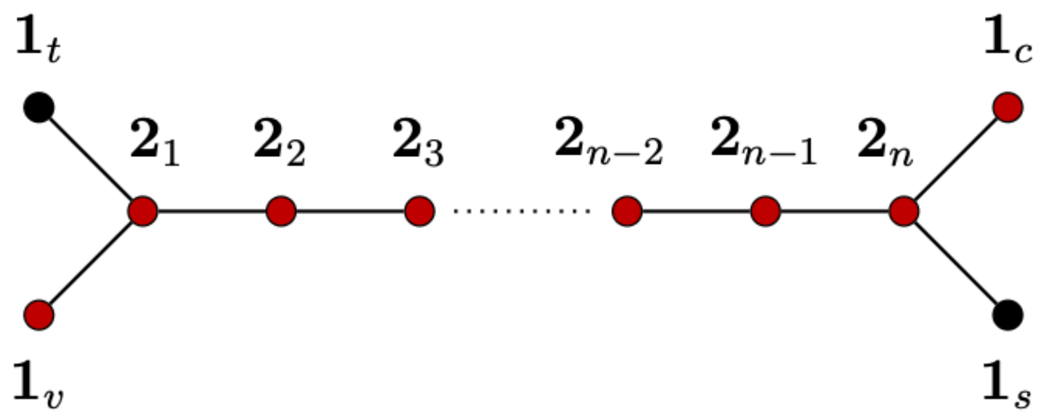


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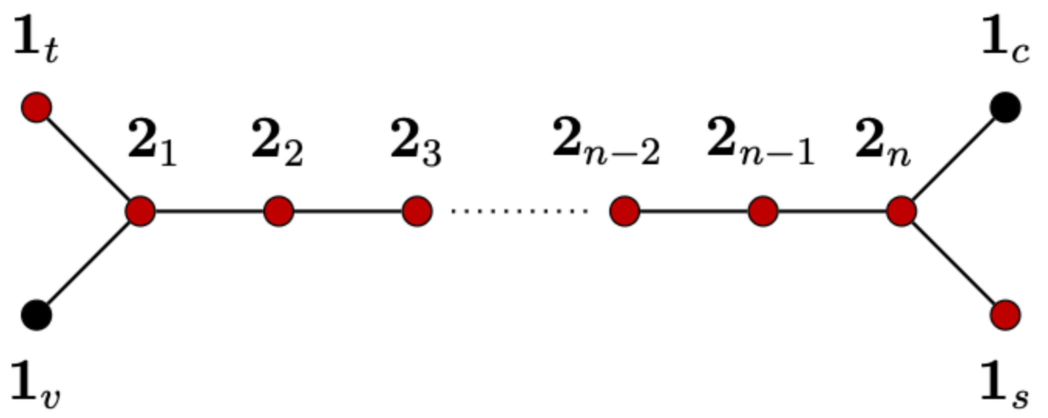


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# Conclusion

- Direct Analysis of Frozen ADE singularities in IIA
- K-Theory + RR-Fluxes + Orbi-Circle Bundles
- Freezing as Confinement

# Outlook

- Freezing as a stabilizing mechanism for Non-SUSY singularities?
- Freezing CY3-fold singularities and others
- Compact Models and ``The Permafrost Landscape” [Morrison, Sung; 2024]